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A DYNAMIC TRANSPORTATION SUB-SYSTEM ANALYSIS FOR OPTIMIZING A MATERIAL HANDLING SYSTEM WITH MULTIPLE TRANSFERS

by

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INTRODUCTION

The decision maker is often forced to decide between obscure or undefined alternatives. This dilemma is particularly acute when the number of alternatives is extremely large. For example, an executive who is bidding on a project to install a continuous pipeline through several scattered towns has the problem of selecting one of many possible routes linking them. To avoid being underbid by competitors, the executive attempts to select the shortest route on which to base his costs. In this case, the dilemma he faces is that of selecting the shortest route from a very large number of alternative routes. For instance, if the executive were bidding on a project to install a pipeline through 31 towns, he would need to search through more than 232,200,000,000,000,000,000, 000,000,000,000 possible routes -- a job much too large for the fastest electronic computer.

Traditionally, the routing problem has been illustrated by describing the task of selecting the optimum route for a traveling salesman who starts from a given city and stops at each city of a specified group before returning to his origin. Consequently, though sometimes known as the assignment problem, it has become internationally known as the "traveling salesman problem."

In 1959 Kaufmann (11) described the difficulty of solving this problem:

> We regret to state that there is, at present, no analytical method that makes it possible, in the general case, to find the minimum solution of a traveling salesman problem, other than by trying all the permutations, whose number quickly becomes astronomical: for example, with symmetrical unit costs:
>
> 10 cities (transfers): 181,940 cycles (routes)
>
> 20 cities: a number of 19 digits.
>
> Even the fastest electronic computer would never complete such a task.

Merrill M. Flood (8) stated in 1955 that there were no computational methods, and surprisingly few mathematical results, relative to the problem. At about the same time, G. Dantzig, R. Fulkerson, and S. Johnson (6) agreed that little was known, and that their method of using combinatorial analysis could not be used as a routine procedure. The fundamental question they raised was: in general, does the use of a few linear inequalities reduce the combinatorial magnitude of such problems significantly? Their answer was: "We do not know the answer to this question in any theoretical sense. . ." In 1958, G. A. Croes stated that past efforts to find an efficient solution met with only partial success. In 1963, Little, Murty, Sweeney, and Karel (14) commented:

> In recent years a number of methods for solving the problem have been put forward. Some suffer from inefficiency, others produce solutions that are not necessarily optimal, and still others require intuitive judgments that would be hard to program on a computer.

Mathematicians apparently discussed the optimum route problem informally at meetings for many years: Hassler Whitney raised the question at a Princeton seminar in 1934, according to Flood (8), who created widespread interest when he attempted to optimize the routing of school buses as early as 1937. Soon after 1950, he joined Robinson, Koopmans, Beck, Heller, and Kuhn in exploring the relationship between this problem and the linear programming (transportation) problem.

It is interesting, however, that few results were published before 1954. Both Dantzig (7) and Flood (8) observed the similarity with the so-called Hamiltonian game (concerned with finding the number of different tours possible over a specific network), and discussed the possibility that it stimulated investigation of the shortest tour problem. In 1954, Dantzig, Fulkerson and Johnson modified the linear programming algorithm and applied it to a few traveling salesman problems. One year later, G. Morton and A. H. Land (17) believed they had formally stated this problem in linear programming terms. They felt they had avoided one of the major difficulties in linear programming formulation (the appearance of sub-cycles) by including a time subscript in each distance notation, thus making the program dynamic. (However, in a discussion which followed, Dr. I. Heller stated that this "dynamic" approach was not the linear programming form of the problem.) In 1957, G. A. Croes (4) presented a method that successively improves a given route until certain specified improvements are exhausted. Also in 1957, Minty (15) described his analog string model, and Barachet (1) published his graphic approach. In 1963, R. L. Karg and G. L. Thompson (10) employed a heuristic approach, and more recently (1965), Shen Lin (13) introduced two computer programs which were useful only for symmetric routing.

The objective of this investigation is to develop efficient and reliable methods for selecting the optimum route from a very large number of possibilities, and to develop techniques which will permit route men, such as milk haulers, to select the optimum sequence of pickups or deliveries. Both manual and computer approaches will be utilized.

Companies financing pipeline projects are similarly interested in route selection, because they would prefer to ask for bids on the shortest route, thus lowering installation as well as future operating and maintenance costs.

Similar routing problems occur when electric power plants are linked together to insure continuous service, or when television relay stations are interconnected to extend service to new customers. Sequencing problems which also fit this category include the assigning of jobs to machines, the layout of plants and the routing of school buses, pumping station attendants, collection and delivery trucks -- even "paper boys."

REVIEW OF LITERATURE

In 1954, G. Dantzig, R. Fulkerson, and S. Johnson (7) described a linear programming approach that sometimes, at least, enabled one to find an optimal route and prove it so. Then, in 1957, Dantzig (5) presented a simplex algorithm for finding the shortest distance from the initial node of a network to each other node. To accomplish this, he first selected several arbitrary routes which fanned out from the initial node like the branches of a tree. Each of the other nodes had only one link leading to it, and distances along each branch were accumulated and recorded at the nodes. A direct link lies along a branch, but an indirect link does not. When the shortest path between the initial node and any other particular node contains an indirect link, it becomes part of a branch, and the inferior link is eliminated from the branch. Similarly, other links which are not a part of a branch are introduced one at a time, and are included when any node value is decreased. For each new link added to a branch, some other link must be removed; however, a rejected link might rejoin a branch at some later time. This procedure continues until no further decreases are possible, indicating that the "shortest-route tree" has been produced. Pollack and Wiebenson (21) illustrate the "shortest-route tree" pictorially in describing Menty's string model. Other more efficient methods have since been suggested.

In 1955, Flood (8) clarified some of the relationships between traveling salesman, transportation, distribution, and assignment problems. He pointed out that other authors had written on these relationships; for example, Julie Robinson solved the assignment problem while searching for a solution to the traveling salesman problem, and T. C. Koopmans discussed the possible relationships between the traveling salesman and distribution problems. Flood showed how the method for solving the assignment problem might be applied to the traveling salesman problem; however, he did not show how such a procedure could be efficiently used on a variety of actual situations.

L. L. Barachet (1) in 1957 presented a graphic approach, which starts with an arbitrary route and forms new routes by first changing every group of three consecutive segments that improves the route. Then, each improved route is further improved by additional changes of four, five, . . . *n*-1 consecutive segments. The author's method is rather awkward to apply, and he admits that one cannot be certain that the optimum route will be produced, even when no group of *n*-1 consecutive segments can further improve the route.

G. A. Croes (4), in 1958, applied a simple transformation called "inversion", to transform a trial solution into another which has lower costs. He continued modifications until no further inversions seemed desirable, but there was no assurance that the optimum route had been achieved. He developed another method, to be used as a final adjustment, which gives some added assurance of selecting the desired route; but these final adjustment procedures are rather tedious and time consuming if done manually. The author admits that such procedures would be difficult to program on a computer, because they involve mostly inspectional work.

The heuristic approach, proposed by R. L. Karg and G. L. Thompson (10) in 1963, selects a pair of cities at random, and combines them with a third in such a way as to minimize the length of the three-city subroute. Then a fourth city is selected and included, in such a way as to minimize the resulting four-city subroute. Other cities are included in this manner until the route is composed of $\underline{n}$ cities. The generation of each route begins with the random selection of two cities and finishes when all $\underline{n}$ cities have been included in such a way as to minimize the resulting subroutes. This procedure is continued until some arbitrary number of routes has been generated. The best route generated depends on the pair of cities chosen at random, and on the the order in which the remaining cities are selected. According to the authors, the probability that the first generated route is optimum is 0.16 for $\underline{n}$ = 10 and 0.0045 for $\underline{n}$ = 42. Large problems are factored into subproblems, and each subproblem is generated separately in order to reduce the computational effort required. The authors do not claim this method is infallible; they do say that good answers may be attained in relation to the amount of computer time used. This method is not at all satisfactory for the manual approach.

Also in 1963, Little, Murty, Sweeney, and Karel (14) presented a "branch and bound" method, in which they break up a set of all routes into increasingly small subsets ("branching"), and calculate for each subset a lower bound on the length of the tours. Eventually, a subset is found that contains a single tour whose length is less than, or equal to, some lower bound for every tour. This method does extend the size of problem that can be reasonably handled by computer, without using methods special to the particular problem.

In 1964, Gilmore and Gomory (9) approached the sequencing problem by considering a machine with a single real variable $\underline{x}$ which describes its state. More specifically, they describe the problem as follows:

> Jobs $J_1, \ldots, J_N$ are to be sequenced on the machine. Each job requires a starting state A_i and leaves a final state B_i. This means that J_i can be started only when $\underline{x} = A_i$ and, at the completion of the job, $\underline{x} = B_i$. There is a cost, which may represent time or money, etc., for changing the machine state $\underline{x}$ so that the next job may start. The problem is to find the minimal cost sequence for the N jobs.

The authors defined their model in terms of a permutation $\underline{h}$ that minimized $\underline{c(g)}$ without requiring the resultant route to be feasible. Then, by carrying out a series of interchanges, they converted permutation $\underline{h}$ into a route $\underline{g}$. These interchanges -- which must be made in a special sequence in order to produce the minimal route $\underline{g}$ -- were chosen by finding a minimal spanning tree.

Their model is difficult to apply, although V. I. Mudrov (18) did point out that his interger linear programming model could eliminate some of the difficulty.

Also in 1964, Boutwell and Simmons (3) attempted to estimate milk assembly costs without considering where the dairy farms were actually located. They designed their model for one type of road network, and they assumed the milk producers to be distributed uniformly over that network. The model was intended only as an example of how route costs might be estimated under assumed conditions; therefore, it does not qualify for the general case.

Shen Lin (13) recently (1965) introduced two computer programs which were useful for symmetric routing only. The program (which he described as being a slight modification of the one by Held and Karp) is very limited because it cannot be used when there are more than 13 cities. Even though this model has little practical value, it did satisfy a few perfectionists because an optimal solution was guaranteed. The second program is useful for a large number of cities, but an optimum solution is not assured and it is limited to the symmetric case. It does appear to be more efficient than the others reviewed here.

Some interesting analog methods have been applied to the shortest route problem. Minty (15) produced a simple analog for the symmetric case by designing a string model so that knots represented cities and string links between the knots represented road distances. When the initial and final knots are pulled apart, the links that stretch tight comprise the shortest route(s). The shortest-route tree may also be found by attaching a weight to each of the knots and lifting the network by the initial knot, thus allowing the links that stretch tight to form the shortest-route tree.

Bock and Cameron used a similar analogy, as described by Peart et al. (19), when they placed a gas-discharge tube along each link of a matrix. The tube conducted electrical current above a certain critical voltage, which represented the length of a particular link. After all the links were electrically connected, a voltage was applied across the initial and final terminals and increased just enough to reach the minimum total critical level, thus causing the tubes along the shortest route to glow.

Increasing the voltage is analagous to pulling apart the initial and final knots of Minty's string network.

Rapaport and Abramson (22) also described an analog device which uses variable electric timers to simulate distances between cities. They designate one end of the electrical timing circuit as the initial city and the other end as the final city. When the master clock is started, the initial node is energized and all timers (one for each link) leading from it are started. Each of these timers attempts to energize the node with which it is linked, but succeeds only when it is the first to signal the node. When a timer energizes a node, it activates a light representing that particular link. The timers that fail to signal the node first never activate any lights. As new nodes are energized, they in turn start timers, and so on, until either the final city is reached or until all nodes are reached, and all the links making up the shortest-route tree (as defined by Dantzig, reviewed above) will be lighted.

None of these authors produced an efficient manual algorithm; none produced a computer algorithm which was applicable to the general problem, and was both efficient and precise.

INVESTIGATION

A typical route with multiple stops (or transfers) begins at some point such as position 1 in Figure 1, then proceeds to destinations 3, 2, 4, and 5 (perhaps in that order), and returns to the starting point. The numbered positions represent points of transfer or stops and the linking segments represent distances between stops. Table 1 includes segment lengths for the route in Figure 1, as well as segment lengths for all other possible routes. Other concepts -- time, cost, etc. -- can also be considered. For example, segment 1,3 in Table 1 has a value of 26, and segment 1,2 has a value of 30, either of which could be in miles, hours, dollars, or some other measurable value.

The sequence of the segments in Figure 1 is 1 to 3, 3 to 2, 2 to 4, 4 to 5, and 5 to 1, and the route length is 160 (26 + 24 + 40 + 30 + 40 = 160). Other ways of indicating sequence are:

(1)	1,3	3,2	2,4	4,5	5,1	
(2) From	1	3	2	4	5	
To	3	2	4	5	1	
(3)	1	3	2	4	5	1

The objective is to select the sequence of segments which will give the optimum route. Mathematically, the problem may be stated as follows: Given a cost matrix $D = (d_{ij})$, where d_{ij} = the cost of going from position $\underline{i}$ to position $\underline{j}$, $(i, j = 1, 2, . . . , \underline{n})$, find a permutation $P = (i_1, i_2, i_3, . . . , i_n)$ of the intergers from 1 through $\underline{n}$ that optimizes

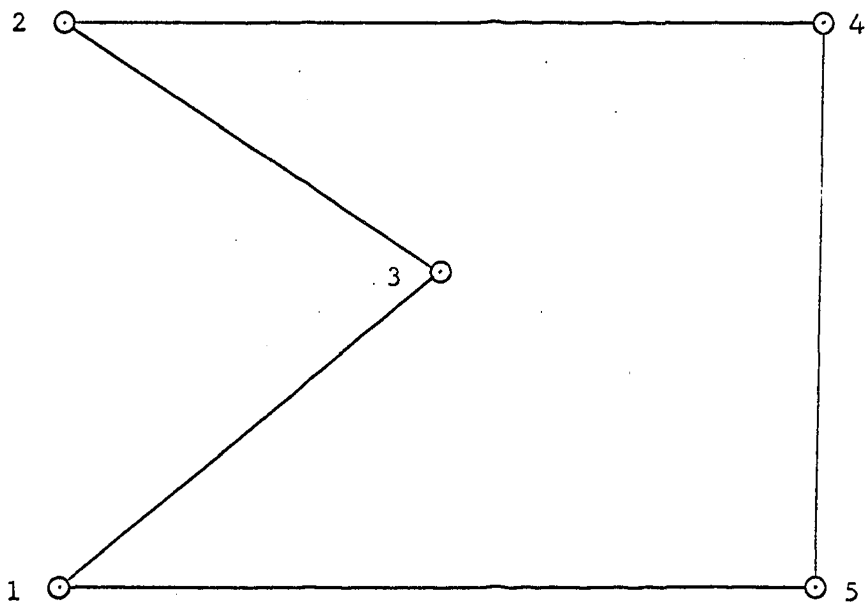


Figure 1. An arbitrary route with five stops (transfers)

Table 1. Cost matrix for a five-position problem

		Route Positions (From)				
		1	2	3	4	5
Route Positions (To)	1	0	30	26	50	40
	2	30	0	24	40	50
	3	26	24	0	24	26
	4	50	40	24	0	30
	5	40	50	26	30	0

the quantity

$$d_{i_1 i_2} + d_{i_2 i_3} + \ldots + d_{i_n i_1} .$$

Since the route in Figure 1 was selected arbitrarily, it may or may not be optimum; however, a later comparison of all 12 possible routes showed that the best route has a length of 148 and a sequence 1 2 3 4 5 1. One difficulty is that as the number of stops increases, the method which measures all possible routes becomes impractical -- even impossible.

One of the objectives of this investigation is to develop efficient and reliable methods for selecting the optimum route from a very large number of possible routes. Since there are numerous ways to attack this problem, the following theses will provide practical guidelines for the investigation:

1. The optimum route may be produced by making particular changes on an arbitrary route.

2. Only feasible changes need to be considered, since other changes produce incomplete routes.

3. Any feasible route may be produced by one or more feasible changes.

4. The optimum route may be selected without considering all possible feasible routes.

Development of efficient methods within these guidelines is still very difficult. One major problem is that of consistently producing feasible changes; another is that of selecting the optimum route from all possible feasible routes. These two major problems appeared to require different approaches; therefore, the investigation proceeded on the following lines:

1. The development of an algorithm which can consistently make changes that produce new and feasible routes; and

2. The development of a second algorithm which can economically select the optimum route from many other feasible routes.

Algorithm for Generating Feasible Routes

During this phase, a great deal of experimenting produced two groups of changes, a feasible group and a nonfeasible group. Analysis of the results showed that every attempt to exchange any <u>one</u> segment on the route for any one not on the route produced a nonfeasible change, and therefore, a nonfeasible route. Also, every attempt to exchange any <u>two</u> segments on a route for any two not on the route failed to produce a feasible change.

However, the following results showed that any <u>three</u> segments may be feasibly changed:

Arbitrary: 1 3 2 4 5

3 2 4 5 1

(26) + (24) + (40) + (30) + (40) = (160)

(Route distances from Table 1)

The resulting modified route is:

Modified: 1 2 3 4 5

2 3 4 5 1

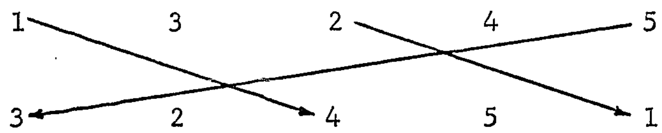
(30) + (24) + (24) + (30) + (40) = (148)

(Route distance)

Note that segments 1 to 3, 3 to 2, and 2 to 4 of the arbitrary route were respectively exchange for segments 1 to 2, 3 to 4, and 2 to 3, and that this particular change was not only feasible, but it produced the optimum (shortest) route.

Any combination of <u>three</u> segments may be feasibly changed, and, as illustrated here they need not be adjacent to each other.

Arbitrary:



Modified:

1	4	5	3	2
4	5	3	2	1

Ten three-segment changes may be made on a five-segment route. This is the number of combinations of three things that can be selected from a group of five, or generally:

$$c = \frac{n!}{r!\,(n-r)!} = \frac{5!}{3!\,2!} = 10$$

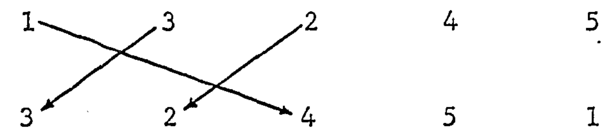
where c = The number of ways of occuring.

n = The number of segments in the group.

r = The number of segments in a change, or the number of things selected from the group.

There is only one way that a particular set of three segments can be changed. Proof of this is easily demonstrated by attempting other changes, such as:

Arbitrary:



Modified:

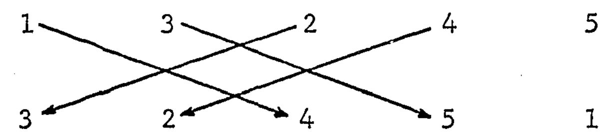
1			4	5
4			5	1

(Nonfeasible)

Note that the modified route is incomplete, because two segments are omitted.

Any <u>four</u> segments may be feasibly changed; as with the three's, there is only one way to feasibly change a particular set of four segments (Shown pictorially in Figure 2):

Arbitrary:

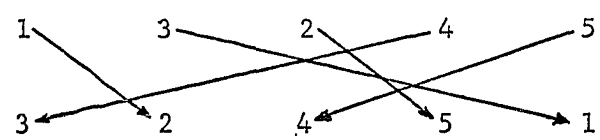


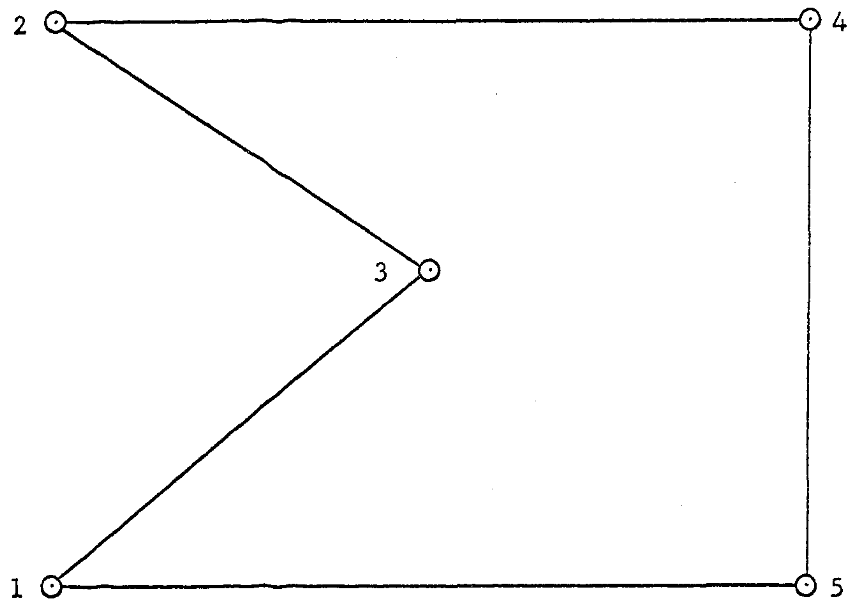
Modified:

1	4	2	3	5
4	2	3	5	1

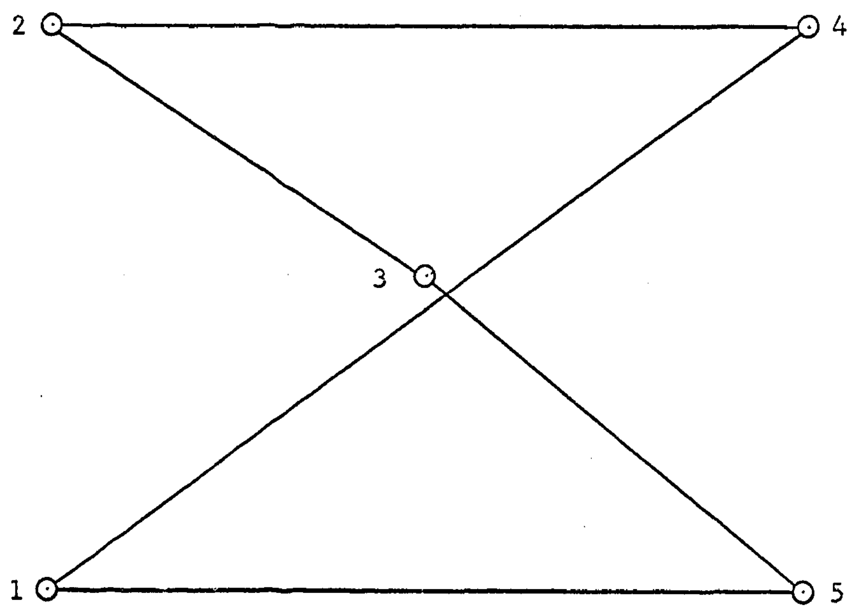
Any <u>five</u> segments may be feasibly changed; however, unlike the three's and four's, a set of five segments can be feasibly changed eight ways, as shown in Figure 3. One of the eight changes follows:

Arbitrary:





Arbitrary



Modified

Figure 2. Effect of a four-segment change

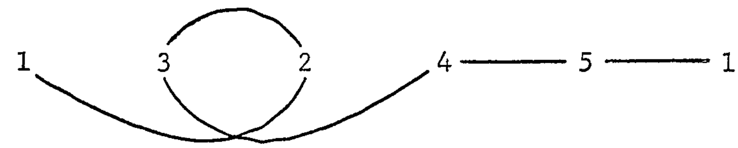
Modified: 1 2 5 4 3

2 5 4 3 1

The investigation also showed that any number of segments greater than two -- every segment of a 1000-segment route, for instance -- can be feasibly changed at one time; but because of the inflexibility of this procedure, it cannot be conveniently adapted to the manual approach.

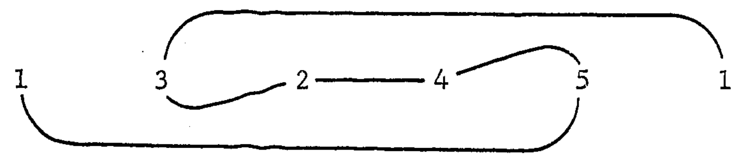
A continued search led to the development of an algorithm which is especially adaptable to the manual approach. Besides being easy to use, this algorithm provides a great deal of flexibility, and an assurance that the changes are actually feasible. Some examples follow:

Arbitrary:



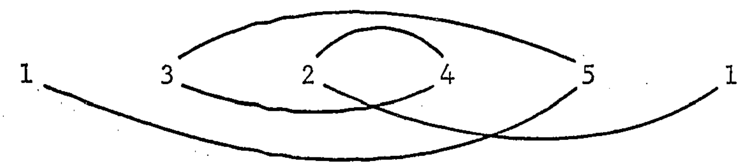
Modified: 1 2 3 4 5 1

Arbitrary:



Modified: 1 5 4 2 3 1

Arbitrary:



Modified: 1 5 3 4 2 1

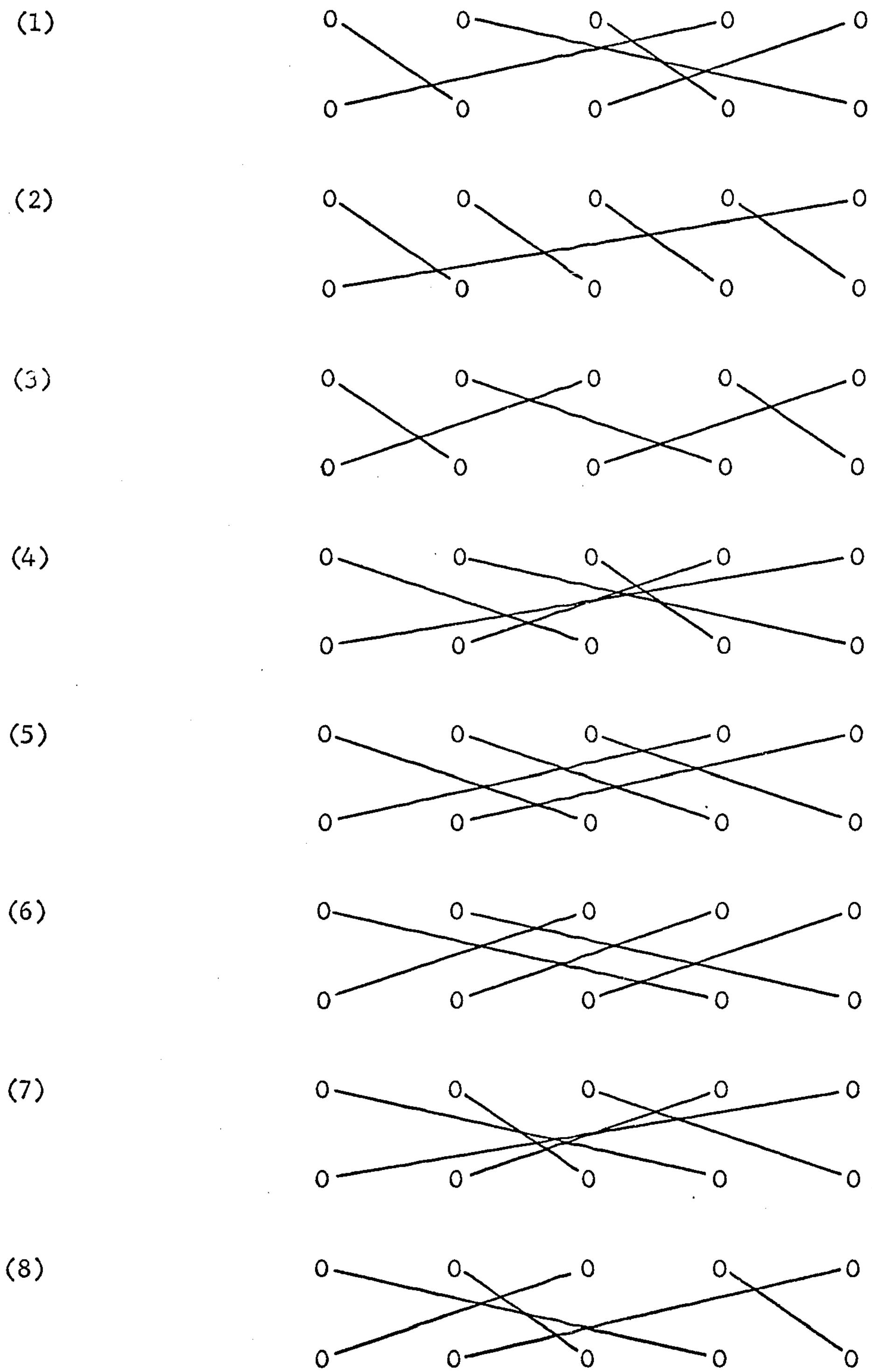


Figure 3. Eight ways to make a five-segment change

Proof that this algorithm can consistently and effectively generate feasible routes is not difficult: the only requirement for a feasible route is that the tour begin at position 1, and end at 1 after passing through each of the other positions only once. One can observe that the changes generated by the new algorithm fully meet this requirement. Now that the generation of feasible changes can be controlled, a procedure is needed to evaluate these changes, so that appropriate ones may be selected and made.

Algorithm for Economically Selecting the Optimum Route

A scheme such as the following, aimed at evaluating the effect of deviating from an arbitrary route, provides one way of developing the algorithm being sought.

1. Segments of the arbitrary route are circled on the cost matrix. See Table 2.

2. Rearrange the cost matrix so that the arbitrary route lies along a major diagonal. See Table 3.

3. Compute the net costs of deviating from the arbitrary route by subtracting each column cost from the circled cost in Table 3, and record the net costs in Table 4.

The circled costs may then be replaced by segment titles, such as 1 to 3 or 1,3, thus eliminating the need for row and column titles. Title 1,3 represents the segment between nodes 1 and 3 -- ie., it represents traveling from city 1 to city 3.

Table 2. Cost matrix for a five-position problem

		Route Positions (From)				
		1	2	3	4	5
	1	0	30	26	50	(40)
Route	2	30	0	(24)	40	50
Positions	3	(26)	24	0	24	26
(To)	4	50	(40)	24	0	30
	5	40	50	26	(30)	0

Table 3. Cost matrix for a five-position problem, rearranged

		Route Positions (From)				
		1	3	2	4	5
	3	(26)	x	24	24	26
Route	2	30	(24)	x	40	50
Positions	4	50	24	(40)	x	30
(To)	5	40	26	50	(30)	x
	1	x	26	30	50	(40)

Table 4. Net cost of deviating from the abritrary route

Route Positions (To)	Route Positions (From): 1	3	2	4	5
3	1 to 3	0	16	6	14
2	-4	3 to 2	0	-10	-10
4	-24	0	2 to 4	0	10
5	-14	-2	-10	4 to 5	0
1	0	-2	10	-20	5 to 1

The route length may be changed by replacing the circled segments with other segments. Positive values in Table 4 shorten the route and negative values lengthen it. For example, to travel from 2 to 3 would shorten the route by 16. The savings of 16 cannot be realized immediately, because one segment cannot be feasibly exchanged for another, and neither can two be exchanged for two others. Therefore, if one wishes to travel from 2 to 3 in order to save 16, he must also travel from 1 to 2 and from 3 to 4. According to Table 4, this change would shorten the route by 12 (16 - 4 + 0 = 12) and produce the following feasible route which is also optimum:

1,2 2,3 3,4 4,5 5,1.

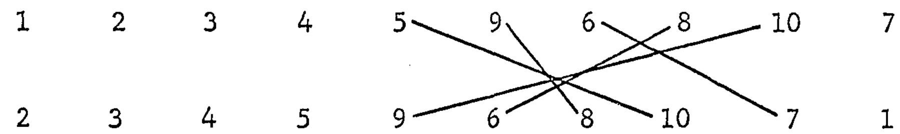
Unfortunately, solutions are not usually so easy. Since there are only 12 possible routes, one would expect an easy solution. However, it is important to note that as the route is lengthened by a few segments, solution becomes tremendously more difficult. Since one objective of this investigation was to discover ways to solve large problems efficiently, three attempts were made to overcome some of the difficulties.

The first attempt called for the selection of an initial, optimum route; but since this was not attained, attention was turned to the selection of an initial route that minimizes the solution effort. A partially successful approach starts the tour at position 1 and always advances it to the next closest available position. Another approach selects several arbitrary routes and tries to transform each into an optimum path. This heuristic method does not guarantee an optimum solution;

and, since precision is dependent upon the number of arbitrary routes selected, one must choose between greater precision and greater economy of selection. Heuristic approaches have been applied to the traveling salesman problem (10), the improved allocation of limited resources on project work (16), and the optimizing of assembly line scheduling (24). Because of the repetitiveness of this method, a computer is usually required.

The second attempt sought for better ways to transform an initial route into the optimum route. Even though it would be desirable to make the transformation instantly, it may not be necessary to do so. In fact, the idea of generating a particular change by combining two or more other changes are explored. For example, two three-segment changes can produce the same reult as a particular five segment change:

(1)



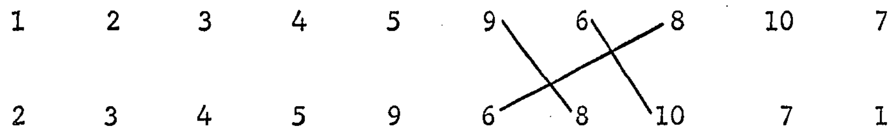
1 2 3 4 5 10 9 8 6 7

2 3 4 5 10 9 8 6 7 1

(Shortest Route)

The same result can also be accomplished by combining two three-segment changes:

(1.a)



(1.b)

1 2 3 4 5 9 8 6 10 7

2 3 4 5 9 8 6 10 7 1

1 2 3 4 5 10 9 8 6 7

2 3 4 5 10 9 8 6 7 1

(Shortest Route)

Also the same result can be attained by making two changes at one time if they are independent of each other:

(2)

1 2 3 4 5 9 6 8 7 10

2 3 4 5 9 6 8 7 10 1

1 2 3 4 5 10 9 8 6 7

2 3 4 5 10 9 8 6 7 1

(Shortest Route)

Another example shows how a seven-segment change can produce the shortest route:

(3)

1 3 2 5 4 9 10 8 6 7

3 2 5 4 9 10 8 6 7 1

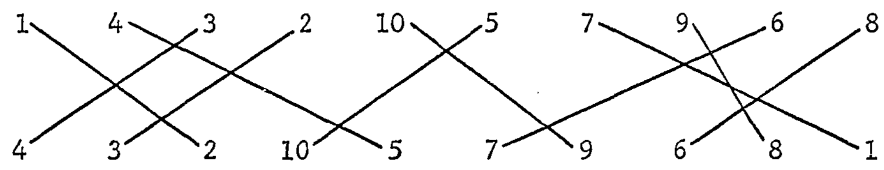
1 2 3 4 5 10 9 8 6 7

2 3 4 5 10 9 8 6 7 1

(Shortest Route)

All 10 segments may be changed at one time to produce the shortest route:

(4)



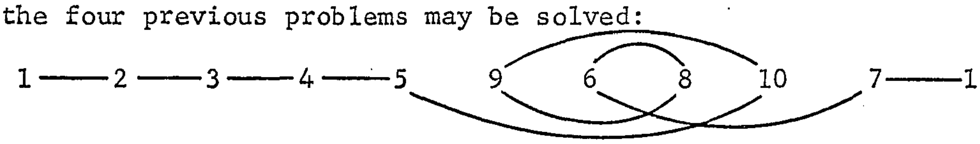
1 2 3 4 5 10 9 8 6 7

2 3 4 5 10 9 8 6 7 1

(Shortest Route)

The third attempt sought to reduce the number of iterative route changes -- without reducing precision, if possible -- and thus reduce the effort required in the search. This attempt successfully developed a procedure which can transform any arbitrary route into the desired route by making all necessary segment exchanges at one time. By this means, the four previous problems may be solved:

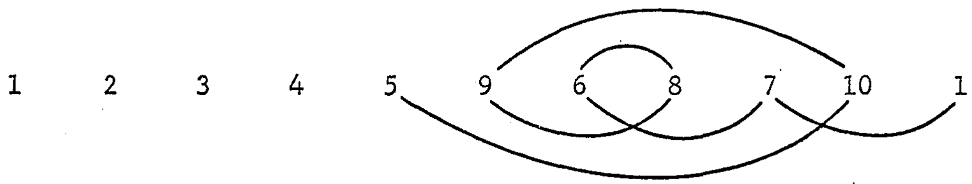
(1)



1 2 3 4 5 10 9 8 6 7 1

(Shortest Route)

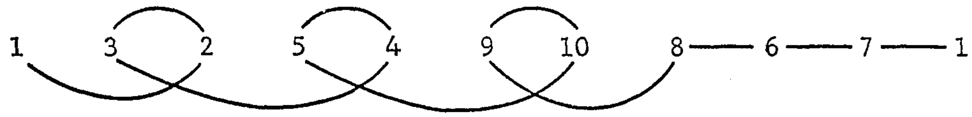
(2)



1 2 3 4 5 10 9 8 6 7 1

(Shortest Route)

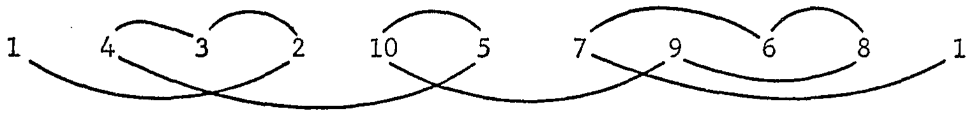
(3)



1 2 3 4 5 10 9 8 6 7 1

(Shortest Route)

(4)



1 2 3 4 5 10 9 8 6 7 1

(Shortest Route)

The <u>manual approach for selecting the shortest route</u> is straightforward and readily understood if the chart (Figure 4, p. 32) is used in two ways. One way (described first) is more effective as a computer approach, and less effective as a manual approach.

After preparing such a chart, three- and/or four-segment changes are evaluated and the best one is selected. Manual effort may be reduced by selecting the most effective combination of independent changes instead of the most effective single change.

After the change or changes have been made, the route sequence is rearranged and the chart is updated. Then the three's and four's are again evaluated, and the cycle is continued as long as the updated route can be shortened. This procedure often produces the shortest route, and in any case, the new route will be shortest or close to it. Greater reliability is ensured by making changes of five, six, seven, etc.; however, the effort required increases rapidly.

Figure 4 shows how a three-segment change is made and evaluated.

Circled numbers designate rows and columns of the original matrix; for example, 1,3 refers to the intersection of column 1 and row 3. The values of segments 1,3 and 1,2 are 26 and 30, respectively, as shown in Table 1. The net cost of using segment 1,2 is -4: 26 - 30 = -4. The other net costs in Figure 4 are computed in a similar manner.

Position numbers defining the arbitrary route are circled in Figure 4. A three-segment change from this arbitrary route is feasible only if each segment's column intersects with another's row at a circle, and if each segment's row intersects with another's column at a circle (See Figure 4). For example, values 6 and -4 intersect at 1,3; values -4 and -2 intersect at 3,2; values -1 and 6 intersect at 4,5. The net cost of this change is 0: 6 - 4 - 2 = 0. If the three-segment change were made, segments 1,3; 3,2; and 4,5 would be exchanged for 1,2; 3,5; and 4,3.

Figure 5 illustrates how a four-segment change is made and evaluated. In this case, four net costs are selected and summed. Starting at some circled number, say 4,5, select a net cost 6 in its column, and another net cost -14 in its row, so that both are equidistant from the circle. Then, starting at another circled number, say 2,4, select a net cost 10 in its column and a net cost 10 in its row, and as before, so that both are equidistant from their common circle. The only other requirement is that both pairs of linking lines must cross, as shown in Figure 5. During this change, segments 1,3; 2,4; 4,5; and 5,1 are exchanged for 1,5; 2,1; 4,3; and 5,4 to shorten the route by 12: -14 + 10 + 6 + 10 = 12. (Note that a four-segment change actually consists of two two-segment changes that overlap.)

Only changes that include at least one positive difference (net cost) need to be evaluated. This reduces the number of permutations to be evaluated, and thus reduces the manual effort. In Figure 4 there are seven possible three-segment changes which have at least one positive difference:

5,4 (10)	+	2,5 (-10)	+	4,1 (-20)	=	-20
5,3 (14)	+	1,2 (-4)	+	3,1 (-2)	=	8
5,3 (14)	+	1,4 (-24)	+	2,1 (10)	=	0
5,3 (14)	+	1,5 (-14)	+	4,1 (-20)	=	-20
5,2 (-10)	+	3,4 (0)	+	2,1 (10)	=	0
4,3 (6)	+	1,2 (-24)	+	3,5 (-10)	=	-28
2,3 (16)	+	1,2 (-4)	+	3,4 (0)	=	12

Of these seven potential changes, two shorten the route, three lengthen it, and two neither lengthen or shorten it. The most effective change would shorten the route by 12. In this particular case, no combination of independent changes can shorten the route more than 12 units.

Feasible four-segment changes are generated in a similar way, and only three of them have at least one positive difference:

5,4 (10)	+	2,1 (10)	+	4,3 (6)	+	1,5 (-14)	=	12
5,1 (-10)	+	3,1 (-2)	+	4,3 (6)	+	1,5 (-14)	=	-20

5,2		3,1		2,3		1,4		
(-10)	+	(-2)	+	(16)	+	(-14)	=	-10

One of the three changes would shorten the route, and the other two would lengthen it. The most effective change shortens the route by 12; as before, no combination of independent changes is more effective.

Next, select the largest change and rearrange the route. The largest change for each algorithm is 12, and the best routes generated by the three's and four's are:

1,2 2,3 3,4 4,5 5,1
(From a three-segment change)

1,5 5,4 4,3 3,2 2,1
(From a four-segment change)

(Note that in this particular case, one of the new routes is the reverse of the other.)

Next, the net cost chart is updated (Figure 6), and new evaluations for the three's and four's are made:

Three-segment changes:

5,4 (10)	+	3,5 (-2)	+	4,1 (-20)	=	-12
5,3 (14)	+	2,4 (-16)	+	3,1 (-2)	=	-4
5,3 (14)	+	2,5 (-26)	+	4,1 (-20)	=	-32
4,3 (6)	±	2,4 (-16)	+	3,5 (-2)	=	-12
5,2 (-10)	+	1,3 (4)	+	2,1 (-6)	=	-12
4,2 (-10)	+	1,3 (4)	+	2,5 (-26)	=	-32

1,3	x	16	6	14
-4	3,2	x	-10	-10
-24	0	2,4	x	10
-14	-2	-10	4,5	x
x	-2	10	-20	5,1

Figure 4. A three-segment change on a net cost chart

1,3	x	16	6	14
-4	3,2	x	-10	-10
-24	0	2,4	x	10
-14	-2	-10	4,5	x
x	-2	10	-20	5,1

Figure 5. A four-segment change on a net cost chart

3,2 (0)	+	1,3 (4)	+	2,4 (-16)	=	-12

Four-segment changes:

5,4 (10)	+	3,1 (-2)	+	4,3 (6)	+	2,5 (-26)	=	-12
5,4 (10)	+	3,1 (-2)	+	4,2 (-10)	+	1,5 (-10)	=	-12
5,3 (14)	+	2,1 (-6)	+	4,2 (-10)	+	1,5 (-10)	=	-12
5,3 (14)	+	2,1 (-6)	+	3,2 (0)	+	1,4 (-20)	=	-12

As just shown, the new route in Figure 6 cannot be further improved with three- or four-segment changes. Actually, it could not have been shortened with any change, because it is already the shortest route. All possible changes have been made to verify that the new route in Figure 6 is the shortest.

This algorithm is more adaptable to the computer approach; however, those wishing to use it manually may reduce the effort required by recording only potentially acceptable changes, and by using overlay guides to quickly locate and evaluate feasible changes.

A more effective algorithm for the manual approach uses a modified net cost chart (Figure 7) which simplifies the selection process, because any feasible route can be generated by extending a line from position 1 through each of the other positions in any order, and finally to position 1 at the other end of the route. The net costs provide a guide for economical sequencing; however, a certain amount of reasoning is required to select the optimum path. For example, according to Figure 7,

(1,2)	0	0	-10	-10
4	(2,3)	0	6	14
-20	-16	(3,4)	0	10
-10	-26	-2	(4,5)	0
	-6	-2	-20	(5,1)

Figure 6. Net costs of deviating from a given five-segment route

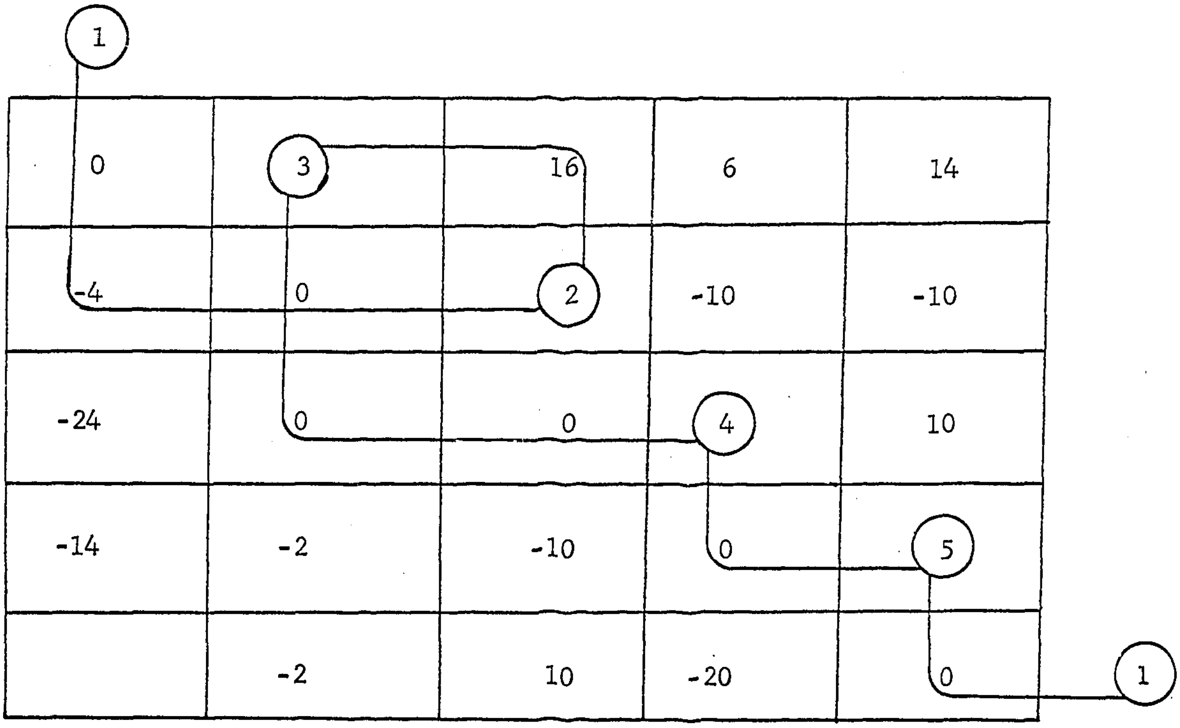


Figure 7. Net costs of deviating from a given five-segment route, and the optimum solution

the biggest single improvement (16) is accomplished by traveling from 2 to 3 instead of from 2 to 4 along the route. For convenient documentation, a line is drawn from 2 to 3 on the chart, and similarly additional lines are completed as other segments are selected. Chart values appearing <u>above</u> the arbitrary route indicate a direction from right to left, and values appearing <u>below</u> indicate a direction from left to right -- e.g., 16 is the improvement of going from 2 to 3, and 0 is the improvement of going from 3 to 2. Position 1 must be linked to another: 1,5 costs 14; 1,4 costs 24; and 1,3 is prohibited because 2,3 has previously been selected; therefore, by process of elimination, segment 1,2 is selected. Number 4 must be linked to another: 4,1 produces an incomplete route, and besides, would cost 10; 4,2 and 4,3 are prohibited because 1,2 and 2,3 were previously selected; so segment 4,5 is selected because of the low cost of 0 and because the other possibilities had already been eliminated. Number 3 must also be linked to another: since 3,1 produces an incomplete route, the only apparent alternative remaining is 3,4. Number 5 can only be linked to 1. The new route (1,2 2,3 3,4 4,5 5,1), as shown earlier, is the shortest path.

Further steps may be taken to verify that the optimum route has been selected:

1. Update the net cost chart with the newest modified route before attempting to find a better route.

2. Make one or more attempts to find the best route without updating the net cost chart. (This step is usually less time consuming.)

A combination of these two steps may be used where reliability is very important, or where a computer can be used for updating. The linking lines as in Figure 7 may be recorded on a transparent overlay for each proposed solution. This procedure is very efficient because no new charts are needed, and it provides a permanent record if desired.

The largest single possible improvement is not always incorporated into the shortest route. This is demonstrated in the solution of the ten-segment problem shown in Table 5. Figure 8 shows the net costs of deviating from an arbitrary route. The largest single improvement is a gain of 56, which occurs when segment 7,3 is exchanged for 7,10. However, it is not selected in this particular case, because to do so would bring about greater losses. Since the route must end at 1 on the right, it is important to evaluate alternative ways to end the route. Segment 7,1 shortens the route by 33, which is by far the greatest end-of-route improvement. All other end-of-route alternatives (except 2 to 1) lengthen the route by 33 or more, so they are not likely prospects. To go from 2 to 1 is not a likely alternative either, because this would prohibit going from 1 to 2, which would cost between 29 and 85. Therefore, segment 7,1 is selected. This selection rules out segment 7,3, which showed the potential improvement of 56. Segment 8,6 is selected, because its potential improvement of 52 is the second largest in the matrix. One segment must end at 8. Segments 7,8 and 6,8 have already been eliminated -- 7,8 because 7 already goes to 1, and 6,8 because 8 already goes to 6. Of the segments available, 10,8 has the least cost, but it should not be used, because other alternatives from 10, such as 10 to 9 or 10 to 5, produce greater savings. So the best alternative appears to be segment 9,8 -- even though it lengthens the route by 12. Now segment

Table 5. Cost matrix for the 10-segment problem

Route Positions (To) \ Route Positions (From)	1	2	3	4	5	6	7	8	9	10
1	0	28	57	72	81	85	80	113	89	80
2	28	0	28	45	54	57	63	85	63	63
3	57	28	0	20	30	28	57	57	40	57
4	72	45	20	0	10	20	72	45	20	45
5	81	54	30	10	0	22	81	41	10	41
6	85	57	28	20	22	0	63	28	28	63
7	80	63	57	72	81	63	0	80	89	113
8	113	85	57	45	41	28	80	0	40	80
9	89	63	40	20	10	28	89	40	0	40
10	80	63	57	45	41	63	113	80	40	0

(1)										
0	(2)	-8	-35	-44	-35	-29	-5	50	17	
-29	0	(3)	-10	-20	-12	0	23	56	23	
-44	-17	0	(4)	0	8	8	35	41	35	
-53	-26	-10	0	(5)	18	6	39	32	39	
-61	-35	-20	-10	0	(9)	0	40	24	40	
-57	-29	-8	-10	-12	0	(6)	52	50	17	
-85	-57	-37	-35	-31	-12	0	(8)	33	0	
-52	-35	-37	-62	-71	-61	-35	0	(7)	-33	
-52	-35	-37	-35	-31	-12	-35	0	0	(10)	
	0	-37	-62	-71	-61	-57	-33	33	0	(1)

Figure 8. Net costs of deviating from a given 10-segment route, and the optimum solution

10,9, with an improvement of 40, can be selected. A segment must end at 10. Of the remaining available segments ending at 10, segment 5,10 is the least costly. The remaining minimum cost segments are 1,2; 2,3; 3,4; and 4,5. Each has a cost of 0. When all of the previous selections are combined the new route becomes:

1	2	3	4	5	10	9	8	6	7
2	3	4	5	10	9	8	6	7	1

(Shortest Route)

This manual approach is easily applied to larger problems, as demonstrated in Figures 9 and 10, where a 26-segment problem is solved. The cost matrix is given in Table 6. Visual selection of the optimum route can be made easier by highlighting the zero and positive values on the net cost chart. (It helped to highlight the zero values with one color and the positive values with another. A third color for the largest value in a column which is zero or larger would also be helpful.)

Some would prefer to display the arbitrary route horizontally, as in the net cost chart in Figure 11. The solution is the same as the one shown in Figure 8.

Computer Approaches for Selecting the Optimum Route

Programming the computer to evaluate all possible routes and select the optimum is very inefficient and impractical, because the total number of routes that need to be evaluated is given by $(\underline{n}-1)!$ for the asymmetrical case and $\frac{1}{2}(\underline{n}-1)!$ for the symmetrical case, where $\underline{n}$ is the number of stops on the route. Therefore, the computer should be programmed to

Table 6. Cost matrix for the 26-segment problem

0	3	4	5	6	3	4	4	5	7	3	5	6	7	7	4	5	6	8	9	6	6	7	8	9	9
3	0	3	3	4	2	3	2	3	5	3	3	4	5	5	5	4	4	6	7	6	5	5	6	7	7
4	3	0	3	3	4	4	3	3	5	5	4	4	5	5	7	6	5	6	7	8	6	6	6	7	7
5	3	3	0	2	4	4	3	1	3	4	4	3	3	3	7	6	4	5	5	8	6	6	5	5	5
6	4	3	2	0	5	5	4	2	3	6	5	4	3	3	8	7	4	5	5	9	7	7	6	5	5
3	2	4	4	5	0	2	3	4	6	2	3	4	5	5	4	3	4	6	7	5	4	5	6	7	7
4	3	4	4	5	2	0	2	4	5	3	2	3	4	4	4	3	3	5	6	6	4	4	5	6	6
4	2	3	3	4	3	2	0	3	5	3	2	3	4	4	5	4	3	5	6	7	5	4	5	6	6
5	3	3	1	2	4	4	3	0	3	5	4	3	3	3	7	6	4	5	5	8	6	6	5	5	5
7	5	5	3	3	6	5	5	3	0	6	5	4	3	2	8	7	5	5	4	9	7	7	6	5	3
3	3	5	4	6	2	3	3	5	6	0	3	4	5	5	3	3	4	6	7	5	4	5	6	7	7
5	3	4	4	5	3	2	2	4	5	3	0	3	4	4	4	3	2	4	5	5	3	3	4	5	5
6	4	4	3	4	4	3	3	3	4	4	3	0	3	3	5	4	2	3	4	6	4	4	3	4	4
7	5	5	3	3	5	4	4	3	3	5	4	3	0	2	6	5	3	3	3	7	5	5	4	3	3
7	5	5	3	3	5	4	4	3	2	5	4	3	2	0	7	6	4	4	4	8	6	6	5	4	4
4	5	7	7	8	4	4	5	7	8	3	4	5	6	7	0	3	4	5	6	3	3	4	5	6	7
5	4	6	6	7	3	3	4	6	7	3	3	4	5	6	3	0	3	4	5	3	2	3	4	5	6
6	4	5	4	4	4	3	3	4	5	4	2	2	3	4	4	3	0	3	4	5	3	3	3	4	4
8	6	6	5	5	6	5	5	5	5	6	4	3	3	4	5	4	3	0	2	5	3	3	2	2	4
9	7	7	5	5	7	6	6	5	4	7	5	4	3	4	6	5	4	2	0	6	4	4	3	2	3
6	6	8	8	9	5	6	7	8	9	5	5	6	7	8	3	3	5	5	6	0	3	3	4	5	8
6	5	6	6	7	4	4	5	6	7	4	3	4	5	6	3	2	3	3	4	3	0	2	3	4	6
7	5	6	6	7	5	4	4	6	7	5	3	4	5	6	4	3	3	3	4	3	2	0	3	4	6
8	6	6	5	6	6	5	5	5	6	6	4	3	4	5	5	4	3	2	3	4	3	3	0	3	5
9	7	7	5	5	7	6	6	5	5	7	5	4	3	4	6	5	4	2	2	5	4	4	3	0	4
9	7	7	5	5	7	6	6	5	3	7	5	4	3	4	7	6	4	4	3	8	6	6	5	4	0

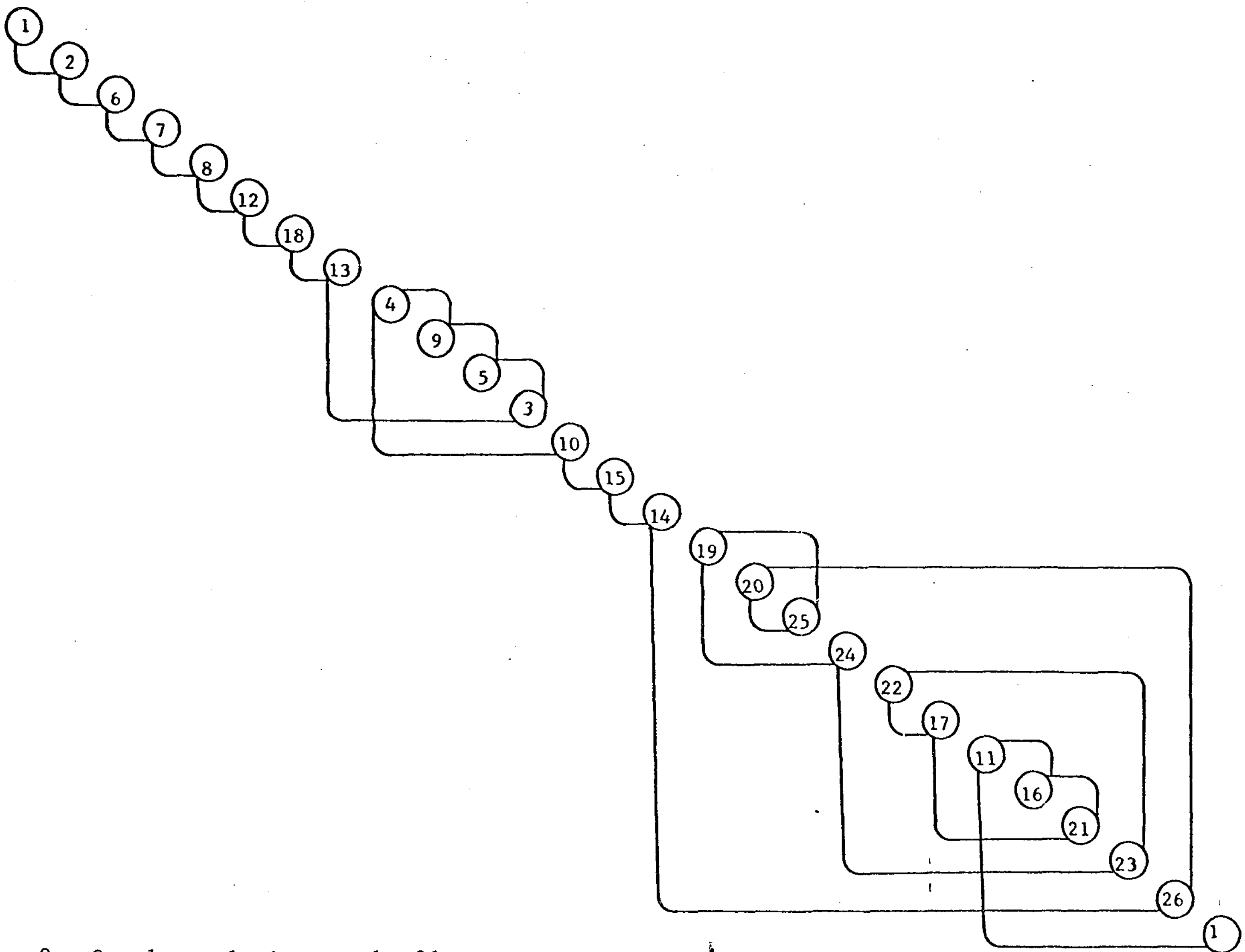


Figure 9. Overlay solution to the 26-segment problem

(1)																										
0	(2)	0	-1	0	-1	-2	-1	-2	-1	-1	2	-3	-3	-2	-4	-5	-4	-3	-3	-1	0	-3	-3	1	2	
0	0	(6)	0	-1	-1	-2	-1	-3	-2	-2	1	-4	-3	-2	-4	-5	-4	-3	-2	0	1	-2	-2	1	2	
-1	-1	0	(7)	0	0	-1	0	-3	-2	-1	1	-3	-2	-1	-3	-4	-3	-2	-2	0	0	-3	-3	2	3	
-1	0	-1	0	(8)	0	-1	0	-2	-1	-1	2	-3	-2	-1	-3	-4	-3	-2	-3	-1	0	-4	-4	2	3	
-2	-1	-1	0	0	(12)	0	0	-3	-2	-2	1	-3	-1	-1	-2	-3	-2	-1	-1	0	0	-2	-2	3	4	
-3	-2	-2	-1	-1	0	(18)	1	-3	-2	-1	0	-3	-2	0	-1	-2	-1	0	-1	0	-1	-2	-2	3	5	
-3	-2	-2	-1	-1	-1	0	(13)	-2	-1	-1	1	-2	0	0	-1	-2	-1	0	-2	-1	-1	-3	-3	2	5	
-2	-1	-2	-2	-1	-2	-2	0	(4)	1	1	2	-1	-1	0	-3	-3	-2	-2	-4	-3	-1	-5	-5	0	4	
-2	-1	-2	-2	-1	-2	-2	0	0	(9)	1	2	-1	-1	0	-3	-3	-2	-2	-4	-3	-2	-5	-5	0	4	
-3	-2	-3	-3	-2	-3	-2	-1	-1	0	(5)	2	-1	-1	0	-3	-3	-2	-3	-5	-4	-3	-6	-6	-1	4	
-1	-1	-2	-2	-1	-2	-3	-1	-2	-1	0	(3)	-3	-3	-2	-4	-5	-4	-3	-4	-3	-2	-5	-5	0	2	
-4	-3	-4	-3	-3	-3	-3	-1	-2	-1	0	0	(10)	0	0	-3	-2	-2	-3	-5	-4	-3	-6	-6	-1	6	
-4	-3	-3	-2	-2	-2	-2	0	-2	-1	0	0	0	(15)	1	-2	-3	-1	-2	-4	-3	-2	-5	-5	0	5	
-2	-3	-3	-2	-2	-2	-1	0	-2	-1	0	0	-1	0	(14)	-1	-1	0	-1	-3	-2	-2	-4	-4	1	6	
-5	-4	-4	-3	-3	-2	-1	0	-4	-3	-2	-1	-3	-2	0	(19)	0	1	1	-1	-1	-3	-2	-2	3	5	
-6	-5	-5	-4	-4	-3	-2	-1	-4	-3	-2	-2	-2	-2	0	0	(20)	1	0	-2	-2	-4	-3	-3	2	6	
-6	-5	-5	-4	-4	-3	-2	-1	-4	-3	-2	-2	-3	-2	0	0	0	(25)	0	-2	-2	-4	-3	-2	2	5	
-5	-4	-4	-3	-3	-2	-1	0	-4	-3	-3	-1	-4	-3	-1	0	-1	0	(24)	-1	-1	-3	-2	-1	3	4	
-3	-3	-2	-2	-3	-1	-1	-1	-5	-4	-4	-1	-5	-4	-2	-1	-2	-1	0	(22)	1	-1	0	0	4	3	
-2	-2	-1	-1	-2	-1	-1	-1	-5	-4	-4	-1	-5	-4	-2	-2	-3	-2	-1	0	(17)	0	0	0	3	3	
0	-1	0	-1	-1	-1	-2	-1	-3	-3	-3	0	-4	-3	-2	-4	-5	-4	-3	-2	0	(11)	0	-2	1	2	
-1	-3	-2	-2	-3	-2	-2	-2	-6	-5	-5	-2	-6	-5	-3	-3	-4	-3	-2	-1	0	0	(16)	0	2	2	
-3	-4	-3	-4	-5	-3	-3	-3	-7	-6	-6	-3	-7	-6	-4	-3	-4	-2	-1	-1	0	-2	0	(21)	3	1	
-4	-3	-3	-2	-2	-1	-1	-1	-5	-4	-4	-1	-5	-4	-2	-1	-4	-1	0	0	0	-2	-1	0	(23)	3	
-6	-5	-5	-4	-4	-3	-2	-1	-4	-3	-2	-2	-1	-2	0	-2	-1	-1	-2	-4	-3	-4	-4	-5	0	(26)	
	-1	-1	-2	-2	-3	-4	-3	-4	-3	-3	1	-5	-5	-4	-6	-7	-6	-5	-3	-2	0	-1	-3	-1	0	(1)

Figure 10. Net costs of deviating from a given 26-segment route

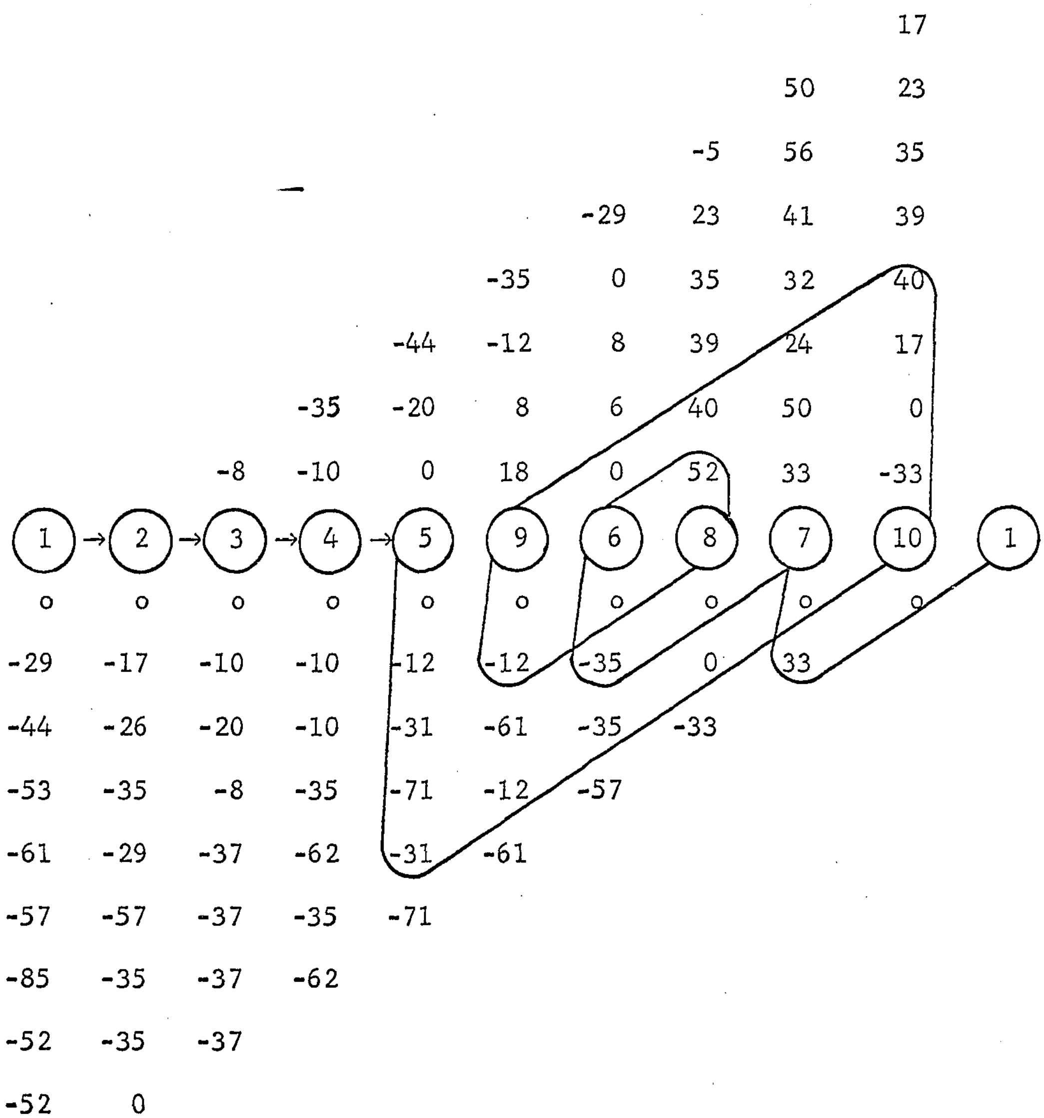


Figure 11. Net costs of deviating from a given 10-segment route

evaluate only a small percentage of the total number of possible routes, or to do the type of reasoning required by the manual approach. Since the computer has not been advanced enough to do this type of reasoning, the investigation was limited to the development of an effective algorithm which evaluates only a small percentage of all routes. As with the manual approach, the thesis of the computer approach was that feasible changes may be made on an arbitrary route in order to transform it into the optimum route.

This investigation started with very basic computer manipulations, and continued through more complex phases. For convenient illustration of the methods investigated, the five-segment problem as shown in Table 2 was analyzed first.

<u>Phase 1</u> The first phase was concerned only with selecting the initial (arbitrary) route. Since any one of ($\underline{n}$-1)! routes may be chosen, it is important to know whether some of them are more likely to produce the optimum route and/or to permit a more efficient solution. The investigation attempted to evaluate a few methods for selecting the initial route and any subsequent arbitrary routes:

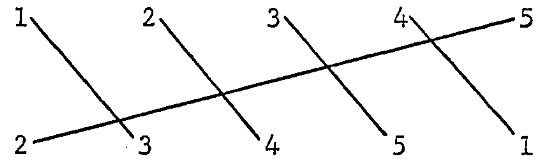
1. Select the major diagonal of the cost matrix for the initial route, i.e., the positions going sequentially from 1 to 2, 2 to 3, 3 to 4, etc.

2. Select the next closest available position, beginning at 1, and returning to 1 after each other position has been included in the route.

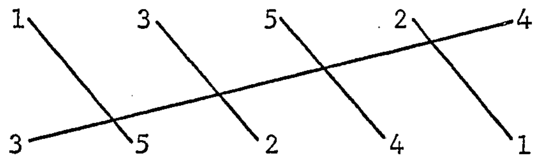
3. Choose positions at random. (This method seems practical when several subsequent arbitrary routes are needed.)

4. Choose any initial route and modify it to produce a second; modify the second to produce a third; modify the third to produce a fourth; etc. For example, arbitrary routes may be generated in the following manner:

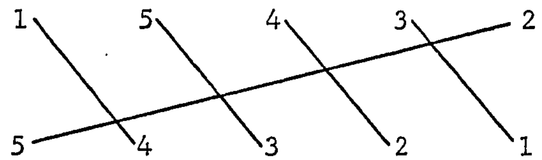
First Arbitrary Route:



Second Arbitrary Route:



Third Arbitrary Route:



Fourth Arbitrary Route:

1	4	2	5	3
4	2	5	3	1

The method of selecting the next closest position was used for much of the remaining investigation. It was initially chosen because it appeared to have two advantages, especially for the manual approach:

1. Initial routes were sometimes close to optimum in length, but were never close to the maximum possible length;

2. Often a front section of the route was optimum, thus forcing unwanted segments toward the end of the route.

Further investigation showed that (1) above had little advantage. For example, the initial route of the 26-segment problem (Table 6) was only one unit longer than optimum, yet 11 segments had to be exchanged to produce the best route. Thus, a route nearly optimum in length is not necessarily easily made optimum. The second advantage was partly confirmed as shown in Figures 8 and 9.

When two or more segments qualify as the next closest position, either one may be selected. For the manual method, the first was usually selected.

Phase 2 The investigation now turns to the generation and evaluation of new routes. Each time a change is made, a new route is generated.

The number of three-segment changes that can be made on an _n_-segment route is given by $\frac{n!}{3!\,(n-3)!}$. For the five-segment route there are $\frac{5!}{3!\,(5-3)!}$ or 10 possible three-segment changes; this agrees with the results in Figure 12.

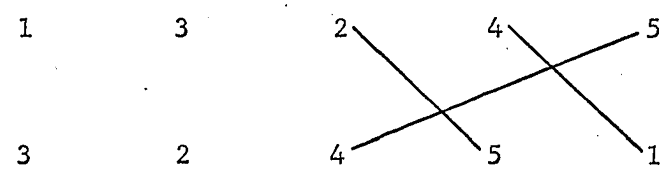
Such changes may be evaluated by two different methods:

1. Evaluate one proposed change and decide whether to make it. If the change is not made, evaluate another and decide whether to make it. When a change is made, the new route is rearranged, and evaluations are again made. This cycle is continued until all three-segment improvements are exhausted.

2. Evaluate all possible three-segment changes on the first route and choose the best one, provided an improvement is made. After the new

route is rearranged, all possible changes are again evaluated and the best one is chosen, provided an improvement is made. After the newest route is rearranged, all possible changes are again evaluated and the one with the greatest improvement is chosen. This cycle is continued until all three-segment improvements are exhausted.

Changes were generated by method 1 in the order shown in Fig. 12 and evaluated:



(Arbitrary route)

1	3	2	5	4
3	2	5	4	1

(Modified route)

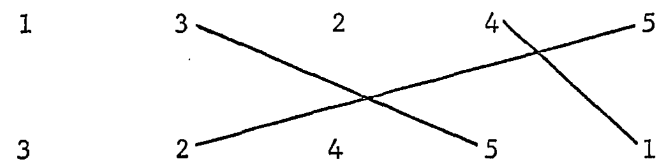
$$
\begin{aligned}
V_c &= (V_{2,4} + V_{4,5} + V_{5,1}) - (V_{2,5} + V_{4,1} + V_{5,4}) \\
&= (40 + 30 + 40) - (50 + 50 + 30) \\
&= 110 - 130 = -20
\end{aligned}
$$

where V_c = Value of the change based on costs in Table 3,

$V_{2,4}$ = Value of the arbitrary segment 2,4,

$V_{2,5}$ = Value of the alternate segment 2,5.

Since this particular change would lengthen the route by 20, it should not be made. The computer then moves to the next possible three-segment change and makes a similar evaluation:



(Arbitrary route)

1	3	5	2	4
3	5	2	4	1

(Modified route)

$$V_c = (V_{3,2} + V_{4,5} + V_{5,1}) - (V_{3,5} + V_{4,1} + V_{5,2})$$

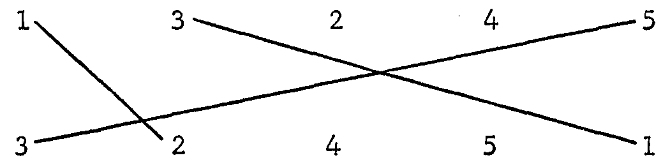
$$= (24 + 30 + 40) - (26 + 50 + 50)$$

$$= 94 - 126 = -32$$

This change would also lengthen the route, and therefore should not be made.

The evaluation of potential changes was continued until the first positive change was encountered, or until all positive changes were exhausted. When the first possible positive change (the sixth in this case) was encountered, it was made:

Arbitrary:



$$(26) + (24) + (40) + (30) + (40) = 160$$

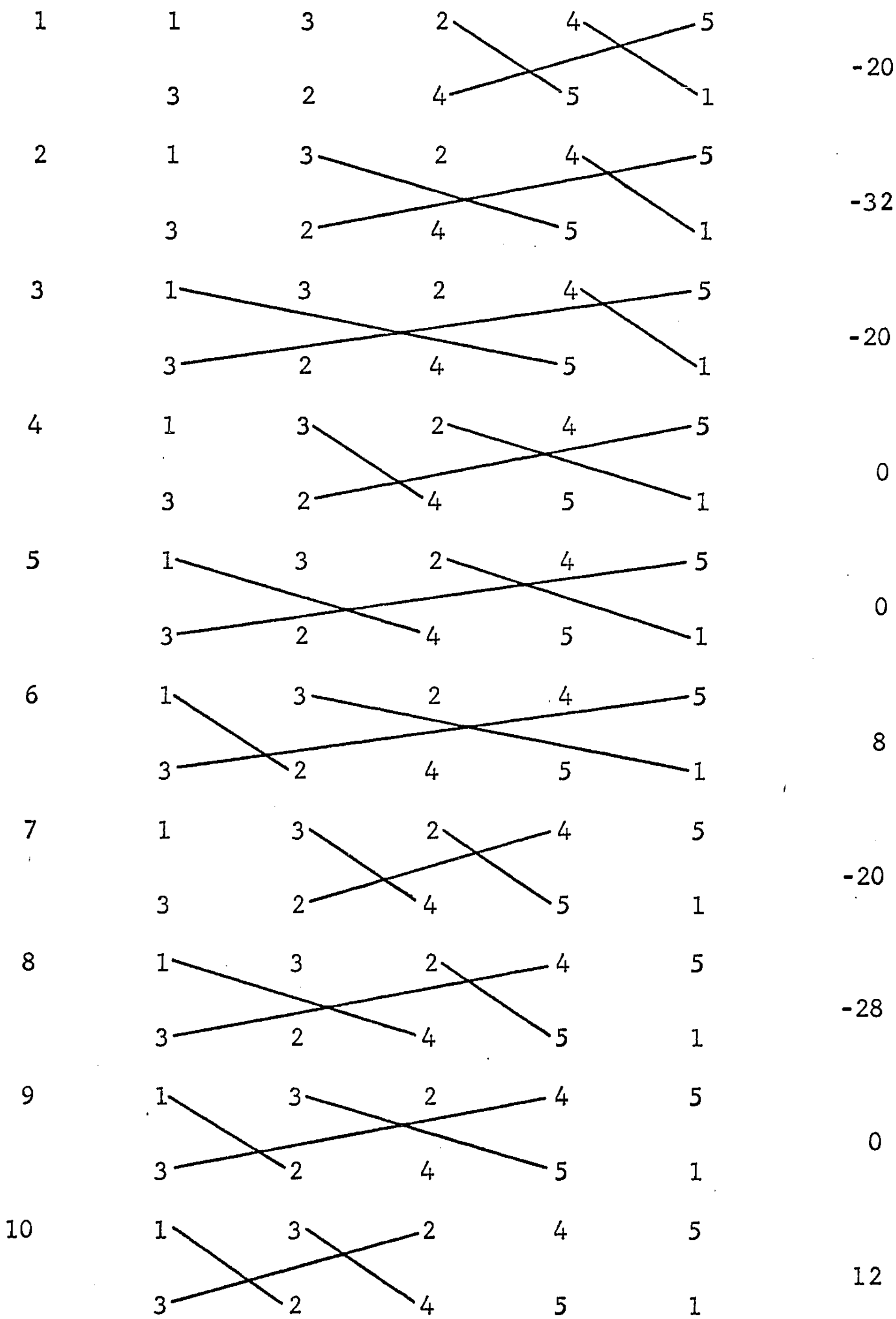


Figure 12. The evaluation of all the 10 possible three-segment changes for the route 1 3 2 4 5 1

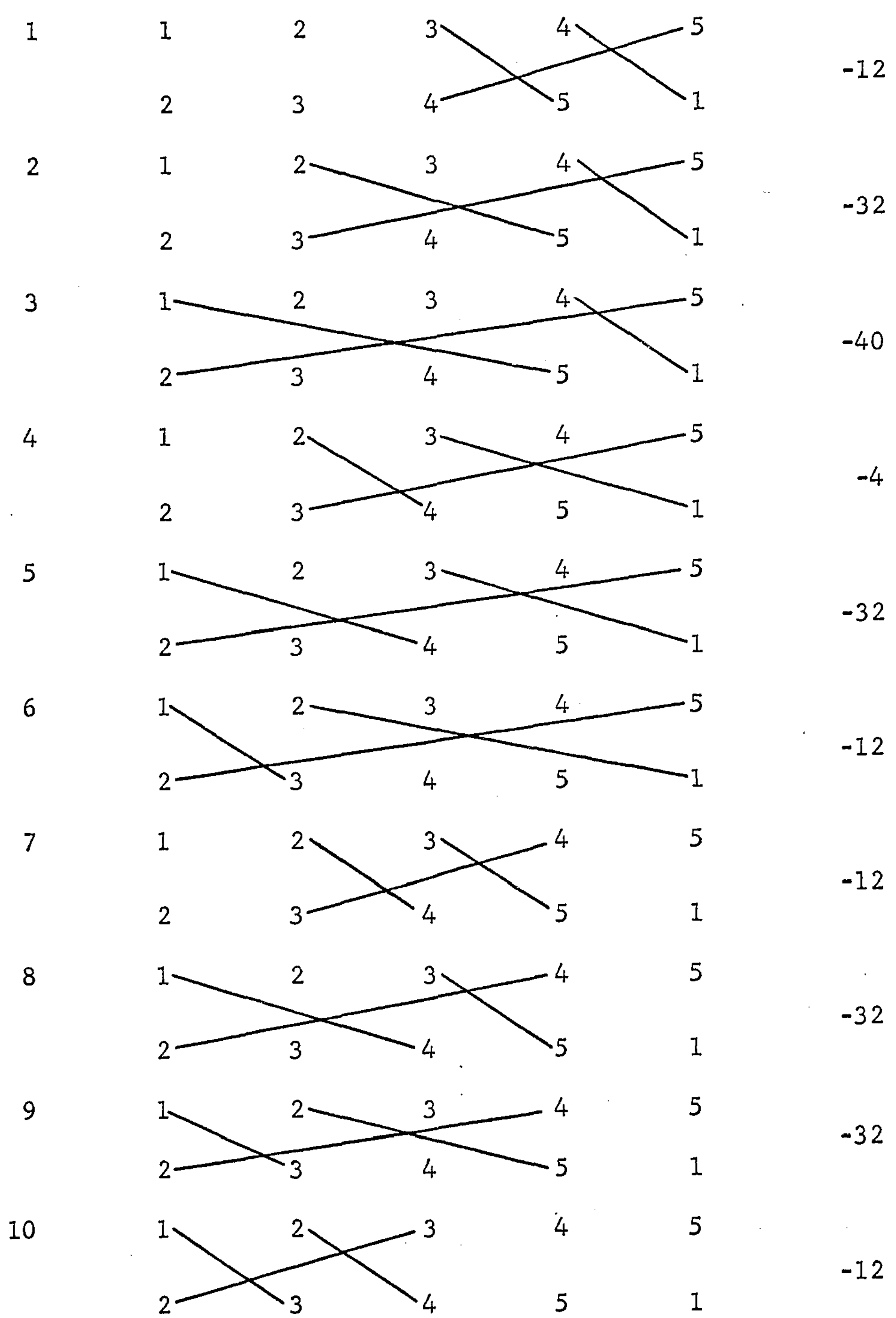


Figure 13. The evaluation of all the 10 possible three-segment changes for the route 1 2 3 4 5 1

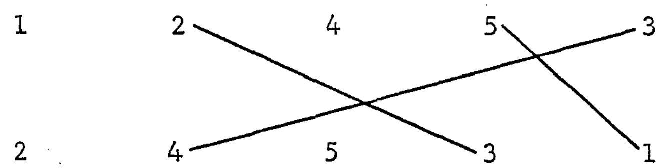
Modified: 1 2 4 5 3

2 4 5 3 1

(30) + (40) + (30) + (26) + (26) = 152

Then, after the modified route was rearranged, new evaluations were made as before, and the first potential positive change was encountered and made:

Arbitrary:



(30) + (40) + (30) + (26) + (26) = 152

Modified: 1 2 3 4 5

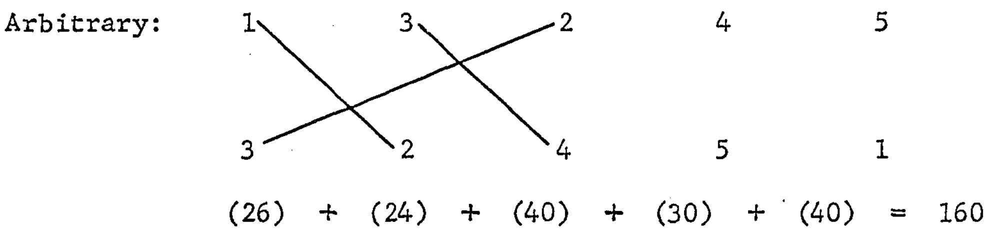
2 3 4 5 1

(30) + (24) + (24) + (30) + (40) = 148

Note that this method produced the optimum route; however, when used in a similar way, it missed the best path for the 10-segment problem by three units.

Optimum results were not attained for either the 15- or 57-segment problems whose cost matrices are presented in the Appendix; but with different initial routes, this procedure <u>could</u> have produced optimum results for both problems. It is easy to see that an initial route could be selected so that a particular number of three-segment changes would produce the optimum length. In practice, however, the task of selecting such an initial route is almost as difficult as selecting the optimum route.

To illustrate the second method, all possible three-segment changes for the five-segment route were evaluated and presented in Figure 12. From these evaluations, the largest possible positive change (12 in this case) was selected to transform the arbitrary route into a new modified route:



Modified:

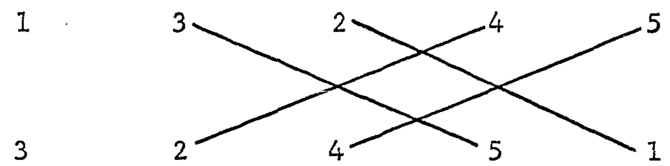
	1	2	3	4	5
	2	3	4	5	1

(30) + (24) + (24) + (30) + (40) = 148

All possible three-segment changes for the new modified route were evaluated (Figure 13). None of the changes further improved the route; however, this was appropriate, since the new modified route was already optimum. This method also produced the optimum route for the 10-segment problem, but missed the optimum for the 26-segment problem by one unit; however, the optimum was attained by modifying the initial route -- the first of two or more segments which qualified for the next closest position was selected, instead of the last.

<u>Phase 3</u> The use of four-segment changes following three-segment changes was evaluated in two different ways: (1) Evaluate all possible changes, select the largest one, and rearrange the new route; perform the same operations on the modified route and on each newly generated route until all positive changes appear to be exhausted. (2) Select the first possible positive change, and rearrange the new route; select the next possible positive change and rearrange the newest route; continue this procedure until all such improvements are exhausted. The four-segment change was evaluated in a way similar to that of the three-segment change. Thus:

Arbitrary:



Modified:	1	3	5	4	2
	3	5	4	2	1

$$V_c = (V_{3,2} + V_{2,4} + V_{4,5} + V_{5,1}) - (V_{3,5} + V_{2,1} + V_{4,2} + V_{5,4})$$
$$= (24 + 40 + 30 + 40) - (26 + 30 + 40 + 30)$$
$$= 134 - 126 = 8.$$

Note that for the symmetric case, only two new segments needed to be compared to two old ones, because the other two new ones were the reverse of the two old ones: $(V_{4,2} + V_{5,4}) - (V_{2,4} + V_{4,5}) = 0$. In this case:

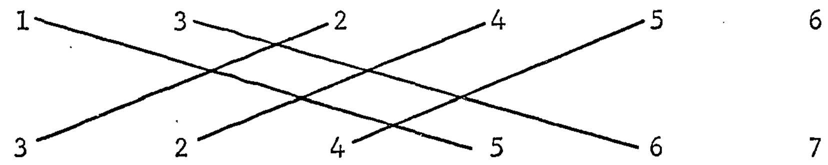
$$V_c = (V_{3,2} + V_{5,1}) - (V_{3,5} + V_{2,1})$$

$$= (24 + 40) - (26 + 30)$$

$$= 64 - 56 = 8.$$

This method was particularly efficient when a large number of segments were evaluated at one time. One type of n-segment change was evaluated by comparing only two new segments with two old ones, as illustrated in the following five-segment change:

Arbitrary:



Modified:	1	5	4	2	3	6
	5	4	2	3	6	7

$$V_c = (V_{1,5} + V_{3,6}) - (V_{1,3} + V_{5,6})$$

The other segments did not change the length for the symmetric case because:

$$V_{2,3} = V_{3,2}$$

$$V_{4,2} = V_{2,4}$$

$$V_{5,4} = V_{4,5} .$$

Phase 4 As shown earlier, two three-segment changes were combined to produce the same result as a five-segment change or a four-segment change. Furthermore, newly generated changes, such as five's and four's, were further combined with three's or others to produce the same results as still larger changes. (The significance of these relationships is that the three's and/or four's can be combined in a particular way to produce the optimum route from perhaps any arbitrary route.)

Several attempts were made to develop a procedure which would combine the appropriate changes required to produce the optimum route.

The first attempt began by evaluating successive three-segment changes as in Phase 2, except that a change was made when the first zero or positive evaluation was encountered. Then, after that particular zero or positive change was selected and made, the route was rearranged. Next, the evaluation of successive three-segment changes was resumed, not at the point where the last change was made, but at the same end of the route where the initial evaluation was made. Again, the first possible zero or positive change was selected and made, and the route was rearranged, as was done previously. This procedure was continued until all possible positive improvements appeared to be exhausted. One difficulty with this procedure was that a zero change could be reversed to reform the original segments, thus causing cycling to occur. This difficulty was overcome by modifying the computer program; however, the solution to the 26-segment problem managed to escape this algorithm also. Using the four's in addition to the three's, (and in a similar manner) did not produce the optimum route either. The program was modified to exhaust all positive three- and/or four-segment changes before any zero changes were made; but even that did not appear to improve the reliability.

Therefore, the first attempt at generating the desired change by combining small changes was not successful. The second attempt, an extension of the first, was developed to give the three's considerably more opportunity to combine and generate the desired results. This approach began by making three-segment evaluations at one end as before and continuing toward the ultimate end until the first zero or positive evaluation was encountered. After the change was selected and made, the route was rearranged as before; but instead of restarting the evaluations at the end of the route, they were restarted at the stopped position. Then, instead of continuing toward the ultimate end of the route in search of zero or positive changes, the reverse direction was temporarily taken. Advancement in the reverse direction was continued as long as any segment of an evaluation overlapped any segment of the initial change (the first change made in the forward direction). At the point of no overlap, the evaluation process reverted back to the point of the initial change and continued toward the ultimate end, until another change was made or until the ultimate end was reached. If another change was made, the reverse direction was again taken temporarily. At the point of no overlap, the evaluation process reverted back to the last change in the forward direction. This cycle was continued until the three's reached the ultimate end.

With this procedure, the optimum route for the 26-segment problem was not found -- nor was the optimum route for the 10-segment problem found.

The third attempt at generating the desired change provided even more opportunity of combining the appropriate changes. In this case, both the three's and four's were used independently in the same manner as the three's were used in the previous attempt. This additional activity not only combined three's with three's, but it combined four's with four's and four's with three's as well. However, this additional feature did not produce the optimum route for the 10- or 26-segment problems. It did, however, produce a nearer optimum route for the 15-segment problem than was produced by the three's alone when searching for the largest positive changes.

Phase 5 The initial route was selected by starting at position 1 and always going to the next closest position. In the case of two or more being equally close, the computer selected the last one evaluated. With this selection, the three-segment algorithm did not find the optimum solution for the 26-segment problem, but did find it when the initial route was formed by choosing the first of equally close positions instead of choosing the last.

This discovery raised another important question: can a more effective and efficient algorithm be developed which evaluates each of several initial routes with a simplified procedure, instead of evaluating one initial route with a complex and lengthy procedure?

In an attempt to answer this question, some fundamental characteristics of the three-segment change were investigated. Answers to the following questions provide an insight into these characteristics:

1. When a change is made, how long are the intervals, or spans, between pairs of adjacent segments? In other words, how long are intervals S_{12} and S_{23} as defined in Figure 14? (The interval between the first

and second segments of the change is represented by S_{12}, and between the second and third by S_{23}.)

2. How do the values of the largest changes vary as successive changes are made?

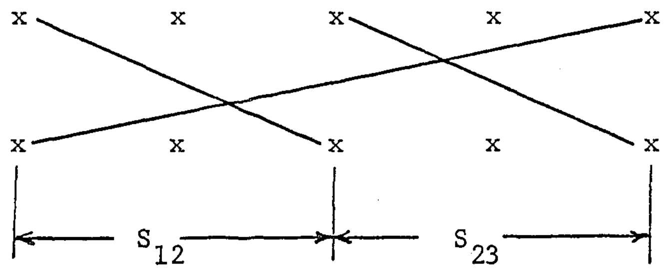


Figure 14. Three-segment change

3. When only the largest possible change is made, how many iterations are required to exhaust all three-segment improvements?

Answers to these three questions were concluded from the experimental results shown in Table 7. In some cases, more results were needed to draw stronger conclusions.

According to Table 7, 14 of the 23 changes had intervals of zero length between the first and second segments, and 11 of the 23 changes had intervals of zero length between the second and third segments. Also, 8 of the 23 changes had intervals of zero length between all segments. Similarly, none of the S_{12} intervals exceeded n/2 and only three S_{23} intervals exceeded n/2. Table 7 also shows that the initial interval lengths are considerably longer than later ones.

Values of the largest change from each iteration are recorded in Table 7. Note that the earlier changes are larger than the later ones. For example, the value of the first change on the 57-segment problem is 1,345, compared to the 13 produced by the last change. The relative decrease in the succeeding values appears to be a function of the variation of segment lengths in the distance matrix. Furthermore, the successive values for the 26-segment problem appears to decrease at a lesser rate than those of the 10- and 57-segment problems. The variation of the segment lengths of the 26-segment problem is small compared to those of the 10- and 57-segment problems. (See Tables 5 and 6 for the distance matrices of the 10- and 26-segment problems.)

Table 7 showed that the number of successive three-segment changes required to exhaust positive changes is approximated by $(\underline{n} + 1)/5$.

The longest possible interval that can occur is $\underline{n} - \underline{r}$, where $\underline{r}$ is the number of segments changed. For example, the longest three-segment interval needed for evaluating the 26-segment problem was given by 23(26-3); yet, according to Table 7, the longest interval for any change actually made was nine in length. Others were much shorter: for example, there were six intervals with zero in length, two with one in length, one with two in length, and one with three in length.

The total number of three-segment evaluations made on the 26-segment problem for the six largest changes is given by:

$$\frac{6n!}{r!\,(n-r)!} = \frac{(6)(26!)}{(3!)(26-3)!} = 15{,}600 \text{ -- or } 2{,}600 \text{ per change.}$$

Table 7. Experimental results of successively generating the largest positive three-segment change on each of several routes

Change Number	Interval Changed S_{12}	Between Segments S_{23}	Improvement Produced by the Largest Possible Change	Number of Segments in Problem, n
1	2	0	42	10
2	0	0	5	
1	2	8	42.4	15
2	0	1	3.1	
3	0	4	3.2	
1	8	4	19.4	16
2	3	0	19.3	
3	0	1	3.6	
1	9	0	40	26
2	3	1	30	
3	0	1	20	
4	0	0	10	
5	0	2	10	
6	0	0	10	
1	15	14	1345	57
2	9	39	315	
3	0	0	207	
4	3	4	192	
5	0	0	98	
6	0	0	92	
7	0	0	39	
8	0	40	13	
9	0	0	13	

If these six changes can be made with interval lengths of nine or shorter, then a procedure which uses interval lengths as long as 23 does a great deal of unproductive searching.

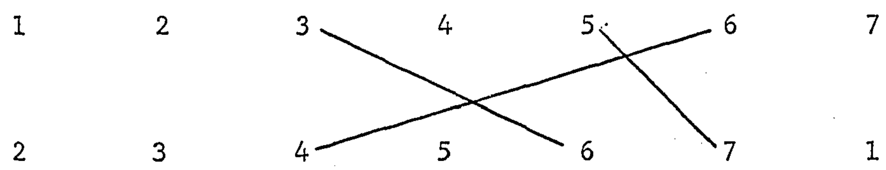
One attempt to eliminate much of the unproductive searching limited the interval lengths to $(\underline{n}+1)/2$. When this restriction was applied to the 26-segment problem, the number of evaluations per change was approximately 1,380, compared to 2,600 without limits. Similarly, since the expected interval links became shorter with later changes, the interval limits were designed to shrink as succeeding changes were made. With limits, this method did efficiently generate the same results as the method without limits for the 10-, 15-, 16-, and 26-segment problems. It did not produce the same results for the 57-segment problem.

One important difference occured when the designed limits prevented any largest change from being made: the loss was not regained during later successive changes. To illustrate this, the second largest change for the 57-segment problem required an interval length of 39 which was greater than $(\underline{n} + 1)/2$, or 29. Table 8 shows that the value of improvements amounted to 2,314 without limits, and to 1,991 with limits.

The best solution was often generated from one initial route, but was more likely from two or more. Such a problem called for a precision versus effort decision: additional initial routes increased the effort, but improved the precision. It also appeared true that a more efficient method of generating and evaluating changes would justify more initial routes. These methods were made more efficient by:

1. Limiting the interval lengths to prevent making unproductive evaluations provided a good example of designing efficient methods. Table 7 showed that four of five problems were solved more efficiently. Failure on the fifth was not as much a drawback as first appeared. When several initial routes were evaluated, not all were adversely affected by the limits. If doubt persists, this method should be followed by another searching technique, which would still permit another opportunity for optimizing the route.

2. Making the first positive change instead of making the largest change led to a more efficient solution. Searching was previously started at one end of the route; after each change was made, it was again started at the same end. On long routes, this was especially unproductive, because searching continued repetitively over portions of the route which had already been improved. This difficulty was eliminated by removing the segments which had been improved and attaching them to the other end of the route:



In this case, evaluations started on the left and proceeded to the right until the first positive change was made as shown above. Previously, searching would have again started with segment 1,2 of the following new route:

Table 8. Effect of limited internal length on generating three-segment changes for a 57-segment route

Change Number	Intervals S_{12}	S_{23}	Improvement from the Largest Change	S_{12}	S_{23}	Improvement from the Largest Change
1	15	14	1,345	15	14	1,345
2	9	39	315	0	0	207
3	0	0	207	1	11	113
4	3	4	192	0	0	98
5	0	0	98	0	0	92
6	0	0	92	0	1	91
7	0	0	39	0	0	39
8	0	40	13	0	0	6
9	0	0	13	-	-	-
Total Route Improvement			2,314			1,991

1	2	3	6	4	5	7
2	3	6	4	5	7	1

However, to permit more efficient searching, the new route was modified to begin with the first segment of the last change:

3	6	4	5	7	1	2
6	4	5	7	1	2	3

Then, when the searching resumed on the left, it was immediately in a fertile area of possible changes. The route was rearranged in this manner after each change. This procedure was designed for any size of change, i.e., three segments, four segments, five segments, etc.

Phase 6 Another algorithm which can readily change any number of segments was developed for the symmetric case. It was easily adapted to the computer approach which was contrary to previous approaches that required a particular computer sub-program for the three's, another for the four's, still another for the five's, etc.

The new pattern of searching (illustrated in Figure 15) made 20 evaluations on a six-segment problem; however, the number in general is given by:

$$\sum_{i=0}^{n-1} \sum_{r=3}^{n-1} (n - r - i + 1)$$

where $\underline{n}$ = The number of segments in the route

$\underline{r}$ = The number of segments evaluated at any one time

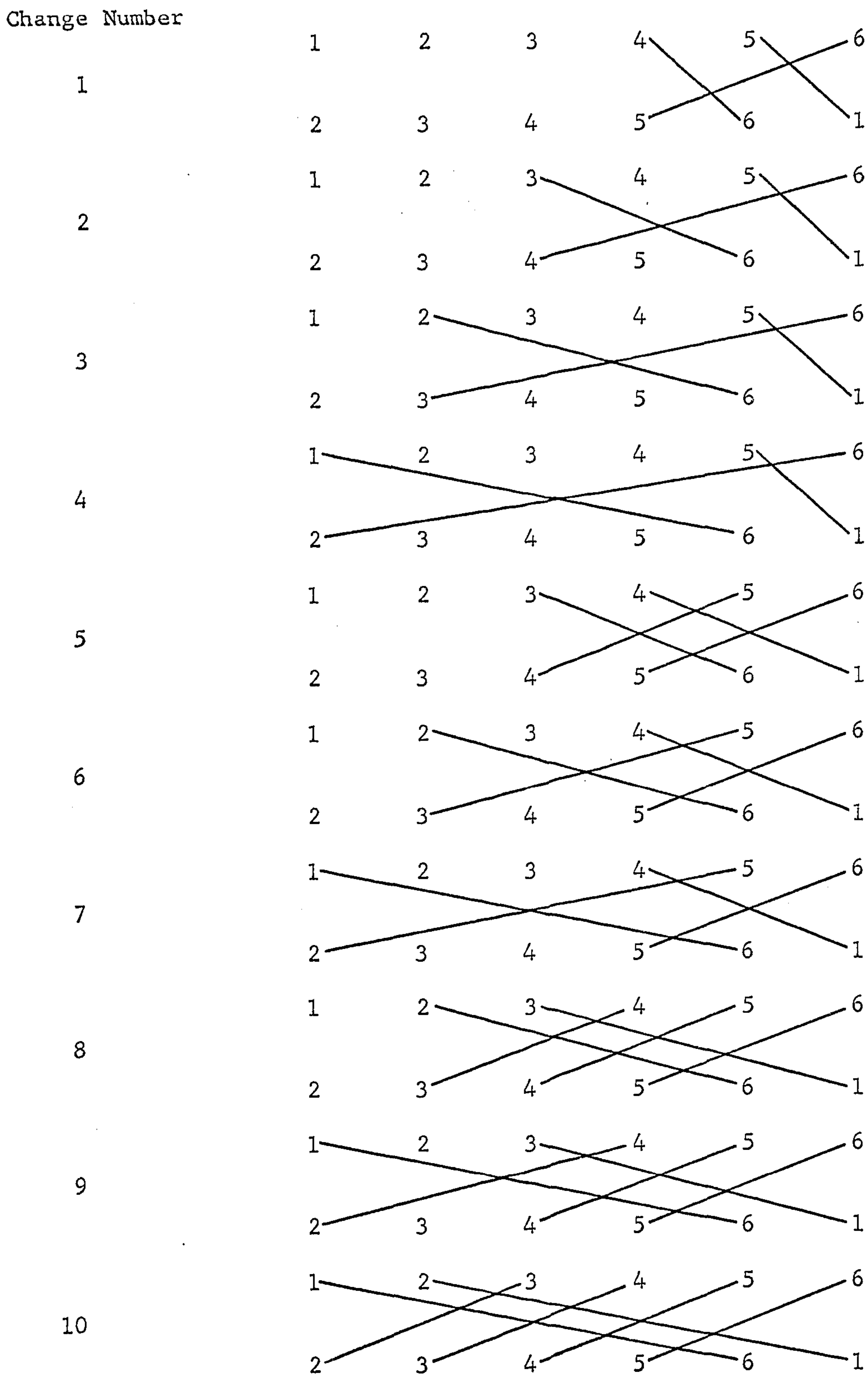
$\underline{i}$ = An indexing variable -- i.e., it moves the evaluation model from one end of the route to the other.

After all possible changes were evaluated (20 in this example), the computer would have selected the largest one and made the change. Then, after this change was made and the route rearranged, all possible changes for the new route were evaluated, and the largest one was again selected and made. This procedure was continued until all positive changes were exhausted.

This algorithm may also be adapted to the method which searches for the first positive change. After the change is made, the computer continues looking for the first positive change. The route may also be modified so that the searching is always limited to unimproved parts of the route illustrated in phase 5.

Phase 7 Computer and manual approaches may be easily combined to provide additional flexibility. For example, after the computer has produced the optimum (or near optimum) route, the manual approach can be used to verify the solution or make improvements if necessary. Since the computer was programmed to print out a net cost chart similar to the one in Figure 8, manual verification can be accomplished within a few minutes.

The computer was also programmed to compute and print out a net cost chart for any route, so that any existing route could be manually improved within a short period of time.


Change Number
1 2 3 4 5 6
1
2 3 4 5 6 1
1 2 3 4 5 6
2
2 3 4 5 6 1
1 2 3 4 5 6
3
2 3 4 5 6 1
1 2 3 4 5 6
4
2 3 4 5 6 1
1 2 3 4 5 6
5
2 3 4 5 6 1
1 2 3 4 5 6
6
2 3 4 5 6 1
1 2 3 4 5 6
7
2 3 4 5 6 1
1 2 3 4 5 6
8
2 3 4 5 6 1
1 2 3 4 5 6
9
2 3 4 5 6 1
1 2 3 4 5 6
10
2 3 4 5 6 1

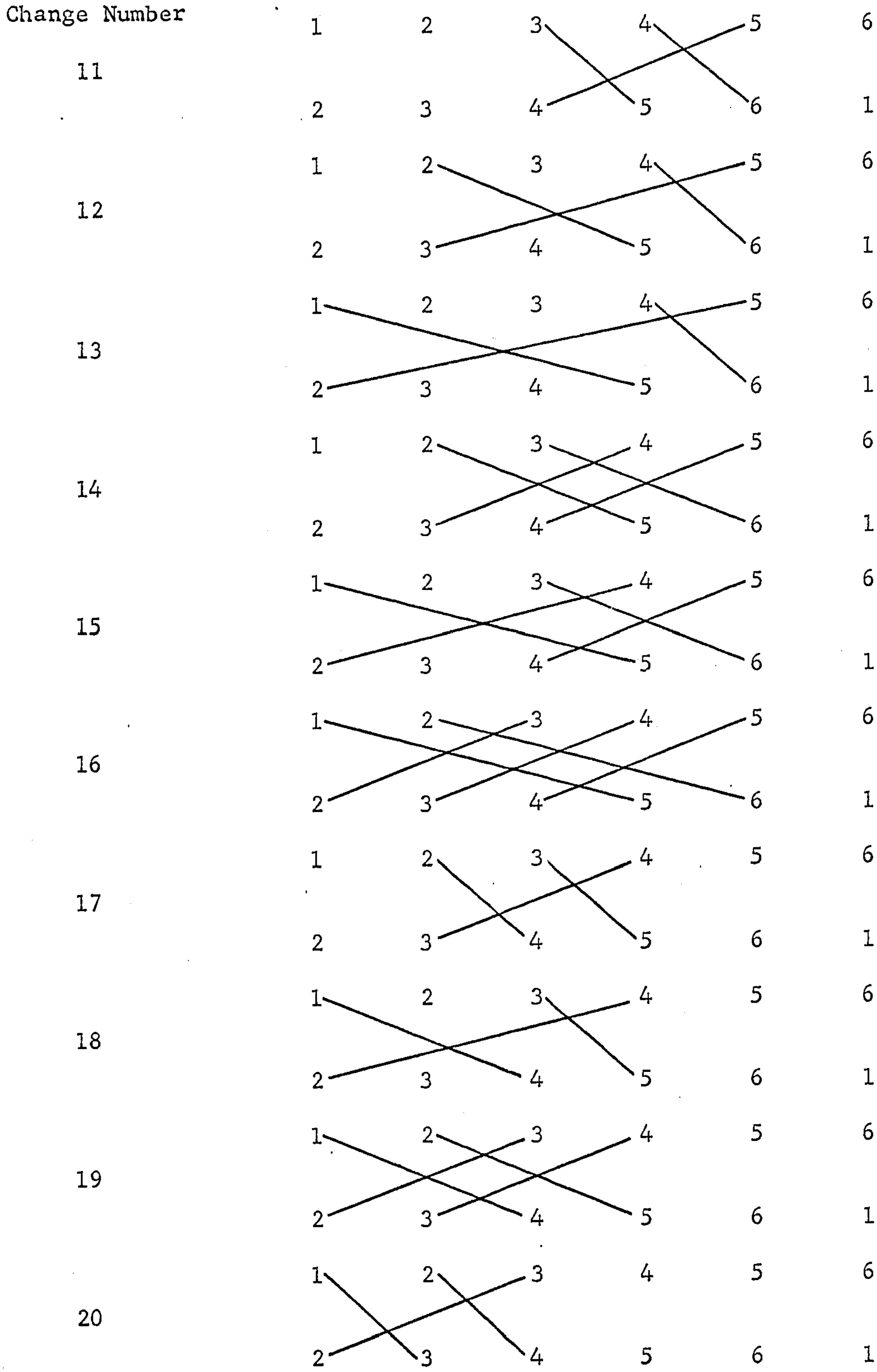


Figure 15 (continued)

APPLYING TRAVELING SALESMAN ALGORITHMS TO MILK COLLECTION ROUTES

Most milk is transported to processing plants in tank-mounted trucks whose hauling capacities range from 1800 to 3300 gallons. Since some trucks travel up to 300 miles per day to collect milk from about 15 dairy farms, it is important that an optimum, or near optimum, route be used. Planning such a route can be a difficult task; for example, there are approximately 1×10^{13} possible routes linking 15 dairy farms -- more than a hauler could ever hope to evaluate. The manual algorithm developed during this investigation can permit haulers to solve such problems. Now that the computer algorithm is available, a group (perhaps a milk cooperative) could determine optimum routes for all haulers.

Three milk routes in central Iowa were selected to illustrate the use of these algorithms. Route A is shown in Table 11, route B in Table 12, and route C in Table 13. Distances were taken from highway commission maps.

Manual algorithm[1] Route A was selected to illustrate the manual approach. Position 1 of the distance matrix represents the truck garage (also the process plant in this case); the other positions represent dairy farms. Route distances actually traveled, which lie along the major diagonal of Table 9, are:

---

[1]No attempt was made to use the computer approach to find the shortest path for Route A; however, the three-segment computer algorithm which searches for zero and positive changes was applied to the initial route once. The computer reduced the distance from 135 miles to 120 miles, and found seven different routes having a length of 120 miles.

Table 9. Matrix of distances from each dairy farm to every other dairy farm, Route A

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	2	8	16	7	8	3	10	11	14	30	31	35	30	30	33
2	2	0	10	18	5	6	4	11	12	16	32	32	37	32	32	34
3	8	10	0	8	8	9	12	14	20	23	29	29	34	29	29	31
4	16	18	8	0	13	13	19	22	27	30	30	29	34	29	28	31
5	7	5	8	13	0	1	7	16	17	20	37	37	43	38	38	40
6	8	6	9	13	1	0	7	18	17	20	37	37	43	38	37	40
7	3	4	12	19	7	7	0	12	12	13	32	33	38	33	33	35
8	10	11	14	22	16	18	12	0	6	9	20	21	26	21	20	23
9	11	12	20	27	17	17	12	6	0	3	21	21	26	21	21	23
10	14	16	23	30	20	20	13	9	3	0	19	20	25	20	20	22
11	30	32	29	30	37	37	32	20	21	19	0	1	6	10	14	15
12	31	32	29	29	37	37	33	21	21	20	1	0	5	9	14	14
13	35	37	34	34	43	43	38	26	26	25	6	5	0	7	12	12
14	30	32	29	29	38	38	33	21	21	20	10	9	7	0	5	5
15	30	32	29	28	38	37	33	20	21	20	14	14	12	5	0	3
16	33	34	31	31	40	40	35	23	23	22	15	14	12	5	3	0

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	1

$$2 + 10 + 8 + 13 + 1 + 7 + 12 + 6 + 3 + 19 + 1 + 5 + 7 + 5 + 3 + 33 = 135$$

(Sum of distances traveled is 135)

After the distance matrix was completed, the next step was to construct a net cost chart. Even though the chart may have been developed from any feasible route, it was prepared from an initial route which was generated by always selecting the next closest available position -- the circled values in Table 9 were selected in this manner. After rearranging the route along the major diagonal, the net cost chart was prepared by subtracting distances not on the route from those on the route (See Figure 16). For example, if one traveled from 1 to 7 instead of from 1 to 2, he would lengthen his journey by 1, as shown in the upper left part of the chart:

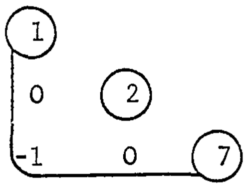
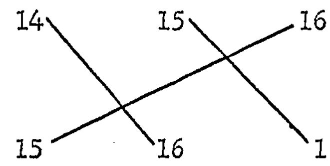


Table 9 also verified this, since the distance 1 to 2 is 2, and the distance 1 to 7 is 3.

The initial route lying along the major diagonal of Figure 16 had a length of 135 miles and a sequence of: 1 2 7 . . . 16 1. It was shortened by changing the sequence of travel; for example, the following change could have reduced the distance by three miles:

Arbitrary:

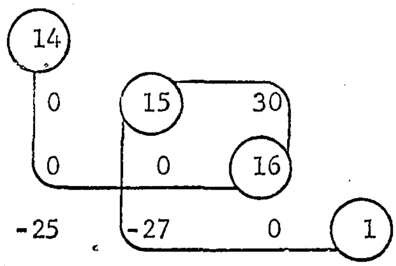


Modified: 14 16 15

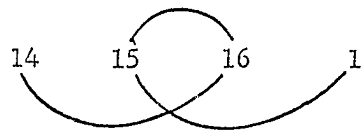
16 15 1

$$\text{Net change} = (V_{14,15} + V_{15,16} + V_{16,1}) - (V_{14,16} + V_{16,15} + V_{15,1})$$
$$= (5 + 3 + 33) - (5 + 3 + 30)$$
$$= 41 - 38 = 3$$

With the net cost chart (Figure 16), the same change was better illustrated:



Old:



New: 14 16 15 1

Net change = (0) + (30) + (-27) = 3

This chart greatly simplified the process of making effective changes on the milk route. The number of changes to be considered was reduced tremendously, because only those which include one or more positive values can shorten the route. Only eight of the 16 columns have positive values and only one value from each column can be selected; therefore, the optimum change can have no more than eight positive values -- the best change had <u>five</u> positive values (Figure 20, p. 77). Of the 239 values in the chart, only 27 are positive. By highlighting these positive values, appropriate changes may be found much more readily.

Using Figure 16, the hauler looks for deviations which improve the route. Not being impressed by the three-mile improvement described earlier, he tries to avoid the -27 by traveling from 15 to 14 instead of from 15 to 1. Using a transparent overlay and a china marker, the hauler draws in a proposed route. He attempts to take advantage of the positive 8 by traveling from 6 to 5 and the positive 14 by traveling from 4 to 3 (Figure 17). These and other improvements would reduce the route distance from 135 to 124 miles. Once such deviations are made and recorded on the chart, other improvements become evident. For example, the use of the positive 14 appears too costly because of having to accept the -6 for segment 3,8 and the -17 for segment 8,16. To avoid these negatives, farm 4 may be linked to farm 16 instead of to farm 3, thus permitting the linking of farm 5 to farm 3 and farm 3 to farm 4. The distance from 5 to 3 and on to 4 is five miles shorter than going directly from 5 to 4. These deviations did improve the arbitrary route by 15 miles (Figure 18). Note that going from 4 to 16 along the new route lengthens the initial route by 9 miles. A possible improvement is to travel from 4 to 15 and on to 16 which would lengthen the initial route by only six miles, -- when extended to another segment, an improvement is verified; for example, the distance from 4 to 14 along this route is 39, as given by:

4		16		15
16		15		14
(31)	+	(3)	+	(5) = 39.
		(Distance)		

Route A: Initial route = 135 miles

(1)																
0	(2)	3	-4	-3	-2	4	-5	-9	3	-31	-27	-30	-27	-29	-1	
-1	0	(7)	-6	2	-4	3	-6	-9	6	-31	-28	-31	-28	-30	-2	
-5	-1	0	(5)	8	0	9	-10	-14	-1	-36	-32	-36	-33	-35	-7	
-6	-2	0	0	(6)	-1	9	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-6	-6	-5	-7	0	(3)	14	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-14	-12	-12	-4	0	(4)	-16	-24	-11	-29	-24	-27	-24	-25	2	
-8	-7	-5	-15	-9	-6	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-8	-5	-16	-8	-12	-5	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-12	-6	-19	-11	-15	-8	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-28	-25	-36	-28	-21	-8	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-28	-26	-36	-28	-21	-7	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-33	-31	-42	-34	-26	-12	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-28	-26	-37	-29	-21	-7	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-28	-26	-37	-28	-21	-6	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-30	-28	-39	-31	-23	-9	-17	-20	-3	-14	-9	-5	0	0	(16)	
	2	4	-6	-1	0	6	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 16. Net costs of deviating from an arbitrary milk route

Route A: Initial route = 135 miles
Improved route = 124 miles

1																
0	**2**	3	-4	-3	-2	4	-5	-9	3	-31	-27	-30	-27	-29	-1	
-1	0	**7**	-6	2	-4	3	-6	-9	6	-31	-28	-31	-28	-30	-2	
-5	-1	0	**5**	8	0	9	-10	-14	-1	-36	-32	-36	-33	-35	-7	
-6	-2	0	0	**6**	-1	9	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-6	-6	-5	-7	0	**3**	14	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-14	-12	-12	-4	0	**4**	-16	-24	-11	-29	-24	-27	-24	-25	2	
-8	-7	-5	-15	-9	-6	0	**8**	-3	10	-19	-16	-19	-16	-17	10	
-9	-8	-5	-16	-8	-12	-5	0	**9**	16	-20	-16	-19	-16	-18	10	
-12	-12	-6	-19	-11	-15	-8	-3	0	**10**	-18	-15	-18	-15	-17	11	
-28	-28	-25	-36	-28	-21	-8	-14	-18	0	**11**	4	1	-5	-11	18	
-29	-28	-26	-36	-28	-21	-7	-15	-18	-1	0	**12**	2	-4	-11	19	
-33	-33	-31	-42	-34	-26	-12	-20	-23	-6	-5	0	**13**	-2	-9	21	
-28	-28	-26	-37	-29	-21	-7	-15	-18	-1	-9	-4	0	**14**	-2	28	
-28	-28	-26	-37	-28	-21	-6	-14	-18	-1	-13	-9	-5	0	**15**	30	
-31	-30	-28	-39	-31	-23	-9	-17	-20	-3	-14	-9	-5	0	0	**16**	
	2	4	-6	-1	0	6	-4	-8	5	-29	-26	-28	-25	-27	0	**1**

Figure 17. Net costs of deviating from an arbitrary milk route, including an improvement of 9 miles

Route A: Initial route = 135 miles
Improved route = 120 miles

1																
0	2	3	-4	-3	-2	4	-5	-9	3	-31	-27	-30	-27	-29	-1	
-1	0	7	-6	2	-4	3	-6	-9	6	-31	-28	-31	-28	-30	-2	
-5	-1	0	5	8	0	9	-10	-14	-1	-36	-32	-36	-33	-35	-7	
-6	-2	0	0	6	-1	9	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-6	-6	-5	-7	0	3	14	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-14	-12	-12	-4	0	4	-16	-24	-11	-29	-24	-27	-24	-25	2	
-8	-7	-5	-15	-9	-6	0	8	-3	10	-19	-16	-19	-16	-17	10	
-9	-8	-5	-16	-8	-12	-5	0	9	16	-20	-16	-19	-16	-18	10	
-12	-12	-6	-19	-11	-15	-8	-3	0	10	-18	-15	-18	-15	-17	11	
-28	-28	-25	-36	-28	-21	-8	-14	-18	0	11	4	1	-5	-11	18	
-29	-28	-26	-36	-28	-21	-7	-15	-18	-1	0	12	2	-4	-11	19	
-33	-33	-31	-42	-34	-26	-12	-20	-23	-6	-5	0	13	-2	-9	21	
-28	-28	-26	-37	-29	-21	-7	-15	-18	-1	-9	-4	0	14	-2	28	
-28	-28	-26	-37	-28	-21	-6	-14	-18	-1	-13	-9	-5	0	15	30	
-31	-30	-28	-39	-31	-23	-9	-17	-20	-3	-14	-9	-5	0	0	16	
	2	4	-6	-1	0	6	-4	-8	5	-29	-26	-28	-25	-27	0	1

Figure 18. Net costs of deviating from an arbitrary milk route, including an improvement of 15 miles

Route A: Initial route = 135 miles
Improved route = 117 miles

1																
0	2	3	-4	-3	-2	4	-5	-9	3	-31	-27	-30	-27	-29	-1	
-1	0	7	-6	2	-4	3	-6	-9	6	-31	-28	-31	-28	-30	-2	
-5	-1	0	5	8	0	9	-10	-14	-1	-36	-32	-36	-33	-35	-7	
-6	-2	0	0	6	-1	9	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-6	-6	-5	-7	0	3	14	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-14	-12	-12	-4	0	4	-16	-24	-11	-29	-24	-27	-24	-25	2	
-8	-7	-5	-15	-9	-6	0	8	-3	10	-19	-16	-19	-16	-17	10	
-9	-8	-5	-16	-8	-12	-5	0	9	16	-20	-16	-19	-16	-18	10	
-12	-12	-6	-19	-11	-15	-8	-3	0	10	-18	-15	-18	-15	-17	11	
-28	-28	-25	-36	-28	-21	-8	-14	-18	0	11	4	1	-5	-11	18	
-29	-28	-26	-36	-28	-21	-7	-15	-18	-1	0	12	2	-4	-11	19	
-33	-33	-31	-42	-34	-26	-12	-20	-23	-6	-5	0	13	-2	-9	21	
-28	-28	-26	-37	-29	-21	-7	-15	-18	-1	-9	-4	0	14	-2	28	
-28	-28	-26	-37	-28	-21	-6	-14	-18	-1	-13	-9	-5	0	15	30	
-31	-30	-28	-39	-31	-23	-9	-17	-20	-3	-14	-9	-5	0	0	16	
	2	4	-6	-1	0	6	-4	-8	5	-29	-26	-28	-25	-27	0	1

Figure 19. Net costs of deviating from an arbitrary milk route, including an improvement of 18 miles

Route A: Initial route = 135 miles
Improved route = 117 miles

(1)																
0	(2)	3	-4	-3	-2	4	-5	-9	3	-31	-27	-30	-27	-29	-1	
-1	0	(7)	-6	2	-4	3	-6	-9	6	-31	-28	-31	-28	-30	-2	
-5	-1	0	(5)	8	0	9	-10	-14	-1	-36	-32	-36	-33	-35	-7	
-6	-2	0	0	(6)	-1	9	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-6	-6	-5	-7	0	(3)	14	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-14	-12	-12	-4	0	(4)	-16	-24	-11	-29	-24	-27	-24	-25	2	
-8	-7	-5	-15	-9	-6	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-8	-5	-16	-8	-12	-5	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-12	-6	-19	-11	-15	-8	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-28	-25	-36	-28	-21	-8	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-28	-26	-36	-28	-21	-7	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-33	-31	-42	-34	-26	-12	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-28	-26	-37	-29	-21	-7	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-28	-26	-37	-28	-21	-6	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-30	-28	-39	-31	-23	-9	-17	-20	-3	-14	-9	-5	0	0	(16)	
	2	4	-6	-1	0	6	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 20. Net costs of deviating from an arbitrary milk route, including an improvement of 18 miles

However, the distance from 4 to 14 along another path is 36, as given by:

4		15		16		
15		16		14		
(28)	+	(3)	+	(5)	=	36.
		(Distance)				

These and other changes did shorten the distance by 18 miles, producing a 117-mile route (Figure 19).

The hauler may be pleased with the shortened distance, but may not be happy with the sequence. Some dairy farmers frequently do their milking late, causing the hauler to either wait or return after collecting from other farms. If position 7 were the farm where the dairyman milks late, the hauler could bypass it and return later (as permitted in Figure 20). This route is still an improvement of 18 miles.

The hauler could improve his current route instead of some arbitrary one, and avoid preparing a second matrix of distances. The generation of improvements from this matrix appears to be a little more difficult; however, it can be just as effective, as will be shown by the changes that follow. These changes also illustrate that the choice of the first improvement is not important because, once it is made, others become immediately apparent. For example, the first improvement discovered on the hauler's route reduced the distance from 135 to 126 miles, a reduction of only nine miles (See Figure 21). As soon as the new route line was recorded, an additional improvement of one mile was discovered (Figure 22).

This procedure of making successive changes was continued until all improvements appeared to be exhausted, when the route distance was reduced to 117 miles. The 18 mile reduction could have been accomplished with only one change, but was not because it is easier to find several successive improvements than to find the single perfect chance. Each adjustment was easily made, since it only required drawing part of a route line on a transparent sheet which covered the net cost chart. Once the new route line was recorded, another transparency was placed over the first one, so that a deviation from the new route could be temporarily recorded and evaluated. After the improved route was recorded on the top transparency, it was then placed directly over the chart so that deviations from it could be evaluated in the same manner. This procedure was continued until improvements appeared to be exhausted.

Table 10 identifies key improvements that reduced the hauler's route length from 135 miles to 117 miles. Figures 21, 22, 23, 24, and 25 show the complete change for each improved route.

Computer Algorithm Position 1 of route B represents the truck garage at the home of the hauler. In the morning, the driver goes directly from his home to the dairy farms, but after collecting the milk he goes to the process plant before returning home. Therefore, the distance from his home to a dairy farm is not the same as from that farm to his home. This is an example of the nonsymmetric case. The models presented by Shen Lin (13) are not appropriate for this type of problem, because they are limited to the symmetric case.

Table 10. Key improvements that reduced the hauler's route length from 135 miles to 117 miles

Change Number	Figure Number	Improvement over: Previous Route	Improvement over: Initial Route	Key Improvements					
1	21	9 mi.	9 mi.						
2	22	1 mi.	10 mi.	Old:	9	1			
				New:	9	8	1		
3	23	2 mi.	12 mi.	Old:	3	16	15	14	
				New:	3	15	16	14	
				Old:	8	1			
				New:	8	2	1		
4	24	2 mi.	14 mi.	Old:	8	15			
				New:	3	4	15		
5	25	4 mi.	18 mi.	Old:	7	6	5	2	3
				New:	7	2	6	5	3

Route A: Initial route = 135 miles
Improved route = 126 miles

(1)																
0	(2)	-2	-5	-4	1	8	-5	-9	3	-31	-27	-30	-27	-29	-1	
-6	0	(3)	5	-7	-2	0	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	8	0	(4)	-12	-5	-7	-16	-24	-11	-29	-24	-27	-24	-25	2	
-5	5	0	0	(5)	6	5	-10	-14	-1	-35	-32	-36	-33	-35	-7	
-6	4	-1	0	0	(6)	5	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-1	6	-4	-6	-6	0	(7)	-6	-9	6	-31	-28	-31	-28	-30	-2	
-8	-1	-6	-9	-15	-11	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-2	-12	-14	-16	-10	0	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-6	-15	-17	-19	-13	-1	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-22	-21	-17	-36	-30	-20	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-22	-21	-16	-36	-30	-21	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-27	-26	-21	-42	-36	-26	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-22	-21	-16	-37	-31	-21	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-22	-21	-15	-37	-30	-21	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-24	-23	-18	-39	-33	-23	-17	-20	-3	-14	-9	-5	0	0	(16)	
	8	0	-3	-6	-1	9	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 21. Net costs of deviating from the milk route traveled by the hauler, including an improvement of 9 miles

Route A: Initial route = 135 miles
Improved route = 125 miles

(1)																
0	(2)	-2	-5	-4	1	8	-5	-9	3	-31	-27	-30	-27	-29	-1	
-6	0	(3)	5	-7	-2	0	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-8	0	(4)	-12	-5	-7	-16	-24	-11	-29	-24	-27	-24	-25	2	
-5	5	0	0	(5)	6	5	-10	-14	-1	-35	-32	-36	-33	-35	-7	
-6	4	-1	0	0	(6)	5	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-1	6	-4	-6	-6	0	(7)	-6	-9	6	-31	-28	-31	-28	-30	-2	
-8	-1	-6	-9	-15	-11	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-2	-12	-14	-16	-10	0	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-6	-15	-17	-19	-13	-1	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-22	-21	-17	-36	-30	-20	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-22	-21	-16	-36	-30	-21	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-27	-26	-21	-42	-36	-26	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-22	-21	-16	-37	-31	-21	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-22	-21	-15	-37	-30	-21	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-24	-23	-18	-39	-33	-23	-17	-20	-3	-14	-9	-5	0	0	(16)	
	8	0	-3	-6	-1	9	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 22. Net costs of deviating from the milk route traveled by the hauler, including an improvement of 10 miles

Route A: Initial route = 135 miles
Improved route = 123 miles

(1)																
0	(2)	-2	-5	-4	1	8	-5	-9	3	-31	-27	-30	-27	-29	-1	
-6	0	(3)	5	-7	-2	0	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-8	0	(4)	-12	-5	-7	-16	-24	-11	-29	-24	-27	-24	-25	2	
-5	5	0	0	(5)	6	5	-10	-14	-1	-35	-32	-36	-33	-35	-7	
-6	4	-1	0	0	(6)	5	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-1	6	-4	-6	-6	0	(7)	-6	-9	6	-31	-28	-31	-28	-30	-2	
-8	-1	-6	-9	-15	-11	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-2	-12	-14	-16	-10	0	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-6	-15	-17	-19	-13	-1	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-22	-21	-17	-36	-30	-20	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-22	-21	-16	-36	-30	-21	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-27	-26	-21	-42	-36	-26	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-22	-21	-16	-37	-31	-21	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-22	-21	-15	-37	-30	-21	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-24	-23	-18	-39	-33	-23	-17	-20	-3	-14	-9	-5	0	0	(16)	
	8	0	-3	-6	-1	9	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 23. Net costs of deviating from the milk route traveled by the hauler, including an improvement of 12 miles

Route A: Initial route = 135 miles
Improved route = 121 miles

(1)																
0	(2)	-2	-5	-4	1	8	-5	-9	3	-31	-27	-30	-27	-29	-1	
-6	0	(3)	5	-7	-2	0	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-8	0	(4)	-12	-5	-7	-16	-24	-11	-29	-24	-27	-24	-25	2	
-5	5	0	0	(5)	6	5	-10	-14	-1	-35	-32	-36	-33	-35	-7	
-6	4	-1	0	0	(6)	5	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-1	6	-4	-6	-6	0	(7)	-6	-9	6	-31	-28	-31	-28	-30	-2	
-8	-1	-6	-9	-15	-11	0	(8)	-3	10	-19	-16	-19	-16	-17	10	
-9	-2	-12	-14	-16	-10	0	0	(9)	16	-20	-16	-19	-16	-18	10	
-12	-6	-15	-17	-19	-13	-1	-3	0	(10)	-18	-15	-18	-15	-17	11	
-28	-22	-21	-17	-36	-30	-20	-14	-18	0	(11)	4	1	-5	-11	18	
-29	-22	-21	-16	-36	-30	-21	-15	-18	-1	0	(12)	2	-4	-11	19	
-33	-27	-26	-21	-42	-36	-26	-20	-23	-6	-5	0	(13)	-2	-9	21	
-28	-22	-21	-16	-37	-31	-21	-15	-18	-1	-9	-4	0	(14)	-2	28	
-28	-22	-21	-15	-37	-30	-21	-14	-18	-1	-13	-9	-5	0	(15)	30	
-31	-24	-23	-18	-39	-33	-23	-17	-20	-3	-14	-9	-5	0	0	(16)	
	8	0	-3	-6	-1	9	-4	-8	5	-29	-26	-28	-25	-27	0	(1)

Figure 24. Net costs of deviating from the milk route traveled by the hauler, including an improvement of 14 miles

Route A: Initial route = 135 miles
Improved route = 117 miles

1																
0	2	-2	-5	-4	1	8	-5	-9	3	-31	-27	-30	-27	-29	-1	
-6	0	3	5	-7	-2	0	-8	-17	-4	-28	-24	-27	-24	-26	2	
-14	-8	0	4	-12	-5	-7	-16	-24	-11	-29	-24	-27	-24	-25	2	
-5	5	0	0	5	6	5	-10	-14	-1	-35	-32	-36	-33	-35	-7	
-6	4	-1	0	0	6	5	-12	-14	-1	-36	-32	-36	-33	-34	-7	
-1	6	-4	-6	-6	0	7	-6	-9	6	-31	-28	-31	-28	-30	-2	
-8	-1	-6	-9	-15	-11	0	8	-3	10	-19	-16	-19	-16	-17	10	
-9	-2	-12	-14	-16	-10	0	0	9	16	-20	-16	-19	-16	-18	10	
-12	-6	-15	-17	-19	-13	-1	-3	0	10	-18	-15	-18	-15	-17	11	
-28	-22	-21	-17	-36	-30	-20	-14	-18	0	11	4	1	-5	-11	18	
-29	-22	-21	-16	-36	-30	-21	-15	-18	-1	0	12	2	-4	-11	19	
-33	-27	-26	-21	-42	-36	-26	-20	-23	-6	-5	0	13	-2	-9	21	
-28	-22	-21	-16	-37	-31	-21	-15	-18	-1	-9	-4	0	14	-2	28	
-28	-22	-21	-15	-37	-30	-21	-14	-18	-1	-13	-9	-5	0	15	30	
-31	-24	-23	-18	-39	-33	-23	-17	-20	-3	-14	-9	-5	0	0	16	
	8	0	-3	-6	-1	9	-4	-8	5	-29	-26	-28	-25	-27	0	1

Figure 25. Net costs of deviating from the milk route traveled by the hauler, including an improvement of 18 miles.

The route actually traveled by the hauler was 253 miles long. His tour, which has a sequence 1 2 3 4 . . . 15 1, lies along the major diagonal of Table 12. The computer selected a different initial route by always proceeding to the next closest available dairy farm. When there were two or more closest farms, the last one was selected. Using this procedure, the initial route chosen was 277.4 miles long. The computer algorithm which searches for the largest three-segment change was used to reduce the route distance from 277.4 to 228.7 miles:

1 11 12 13 14 15 9 8 2 3 7 4 6 5 10

11 12 13 14 15 9 8 2 3 7 4 6 5 10 1

(Initial route distance is 277.4 miles)

1 11 12 9 8 2 3 7 4 6 5 10 13 14 15

11 12 9 8 2 3 7 4 6 5 10 13 14 15 1

(Route distance is 235.0 miles)

1 11 12 9 8 2 3 7 6 5 4 10 13 15 15

11 12 9 8 2 3 7 6 5 4 10 13 14 15 1

(Route distance is 231.9 miles)

1 11 12 9 8 3 7 6 5 4 2 10 13 14 15

11 12 9 8 3 7 6 5 4 2 10 13 14 15 1

(Route distance is 228.7 miles)

Once the distance was reduced to 228.7 miles, further evaluations failed to find another three-segment improvement. When applied after the three-segment algorithm, the four-segment algorithm failed to reduce the distance any further. The complex procedure of generating changes with the three's and four's (described in Phase 4) also reduced the route from 277.4 to 228.7 miles. The computation time on an I.B.M. 360 computer using a Fortran language was approximately 10 seconds.

Route C also has some nonsymmetric distances; for example, the distance from 2 to 1 is 91 miles, while the distance from 1 to 2 is only 37 miles (See Table 13). The route traveled by the hauler was 285 miles long: the computer reduced the distance to 275 with the largest three-segment algorithm, and to 271 miles with the algorithm described in Phase 4. The per cent reduction is deceiving in two ways, since some of the farms are more than a hundred miles (220 round trip) from the process plant, no change short of relocating either the farms or the process plant can improve this situation. The location of the farms form an elongated pattern which makes the task of route selection relatively easy; for example, if all farms were located on a stright line, there would be no question that the shortest route is along the straight line, and the hauler would have no trouble recognizing this. The farms on most milk routes are scattered rather than distributed in an elongated pattern.

DISCUSSION

Obvious uses of the optimum route model were mentioned in the Introduction; however, a use which is not so obvious modifies the shortest route algorithm to give the most reliable route through a network. Let P_{ij} be the reliability of a segment between node _i_ and node _j_, and let P_s be the system reliability. The reliability of a system which has three compartments -- _a_, _b_, and _c_ -- that can fail independently is

$$P_s = (P_a)(P_b)(P_c).$$

Using the relationship:

$$\log_e P_s = \log_e P_a + \log_e P_b + \log_e P_c$$

the segments take on an element of length which can be summed; however, the logarithms of these reliabilities will be negative, because the individual P's are less than unity. Therefore, the more appropriate relationship is

$$-\log_e P_s = -\log_e P_a - \log_e P_b - \log_e P_c.$$

Next, let D be the distance through the network, then

$$D = -\log_e P_s \quad \text{or}$$

$$-D = \log_e P_s. \quad \text{Then,}$$

$$P_s = e^{-D}.$$

The system reliability P_s is equal to e^{-D} where D is the sum of the segment distances between the initial and final nodes of the system network. If there are alternative routes, the maximum reliability is attained when

D is the shortest route. G. R. Shorack (23) describes some procedures for determining the most reliable route.

Another possible application for the optimum route model is in a military strategic or combat situation where there are several objectives to be attained and the cost of attaining any one depends upon which objective was attained immediately before it. In this case, the total cost of accomplishing all the objectives is dependent upon the sequence in which they are attained; therefore, the minimum cost is produced by selecting the optimum sequence (shortest route) through a network of objectives.

In a similar way, if a company's cost of attaining long range objectives depends upon the order in which they are accomplished, the optimum route model can help management do more effective planning and decision making.

This investigation was limited to selecting an optimum route through a given number of points. Further investigation is needed to develop an algorithm which will divide a larger area into two or more optimum routes; for example, when several repair crews service outlying facilities, how should their areas be divided for total optimum cost? Even the number of crews becomes an important factor when the cost of overtime and the cost of overnight lodging are considered. The algorithm may include such factors as the costs of meals and lodging for each night that a crew cannot return home, truck load limitation, overtime wages, and other constraints such as limiting the number of consecutive nights that a crew can stay away from home.

Although effective as a searching tool, each computer algorithm should be investigated further:

1. To determine the precision that results from a given number of evaluations and changes,

2. To find the number of initial routes that should be generated in order to minimize the searching effort for a given precision,

3. To determine how these tools should be combined in order to minimize searching effort for a given precision.

The routing model is also applicable as a search model: One searches for an optimum route just as scientists search for hypotheses, decision-makers for optimal strategies, advertising agencies for customers, and personnel departments for good executives. (The first search model was believed to have been developed during World War II to solve decision problems regarding air patrol searches for enemy submarines.)

SUMMARY AND CONCLUSIONS

The object of this investigation was to develop efficient and reliable methods for selecting the optimum route from a large number of possible routes. Traditionally, this routing problem has been illustrated by describing the task of selecting the optimum route for a traveling salesman who starts from a given city and stops at each city of a specified group before returning to his origin. With a total of (n-1)! routes, the task of evaluating each of them and selecting the optimum one is impractical and often virtually impossible. The number of stops on the route is represented by n.

Efficient manual and computer algorithms capable of transforming a given route into the optimum route were developed. Problems once thought too large for a high-speed computer may now be solved manually and most problems encountered can be solved by a computer algorithm within 30 seconds. These models were used to improve the routing of trucks on milk collection routes having 15 and 16 transfers.

Conclusions:

1. The optimum route may be produced by making particular changes on an arbitrary route.

2. Only one feasible change is required to transform any route into the optimum route, although more than one may be made.

3. The optimum route may be selected without evaluating all possible routes.

4. Only feasible changes need to be considered since other changes produce incomplete routes.

5. A net cost chart which shows the cost of deviating from a given route is a sufficient guide for producing a change that can transform any route into the optimum route.

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APPENDIX

Table 11. Matrix of distances from each dairy farm to every other dairy farm, Route A

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	2	8	16	7	8	3	10	11	14	30	31	35	30	30	33
2	2	0	10	18	5	6	4	11	12	16	32	32	37	32	32	34
3	8	10	0	8	8	9	12	14	20	23	29	29	34	29	29	31
4	16	18	8	0	13	13	19	22	27	30	30	29	34	29	28	31
5	7	5	8	13	0	1	7	16	17	20	37	37	43	38	38	40
6	8	6	9	13	1	0	7	18	17	20	37	37	43	38	37	40
7	3	4	12	19	7	7	0	12	12	13	32	33	38	33	33	35
8	10	11	14	22	16	18	12	0	6	9	20	21	26	21	20	23
9	11	12	20	27	17	17	12	6	0	3	21	21	26	21	21	23
10	14	16	23	30	20	20	13	9	3	0	19	20	25	20	20	22
11	30	32	29	30	37	37	32	20	21	19	0	1	6	10	14	15
12	31	32	29	29	37	37	33	21	21	20	1	0	5	9	14	14
13	35	37	34	34	43	43	38	26	26	25	6	5	0	7	12	12
14	30	32	29	29	38	38	33	21	21	20	10	9	7	0	5	5
15	30	32	29	28	38	37	33	20	21	20	14	14	12	5	0	3
16	33	34	31	31	40	40	35	23	23	22	15	14	12	5	3	0

Table 12. Matrix of distances from each dairy farm to every other dairy farm, Route B

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	125	127	138	143	140	134	116	112	125	70	68	63	58	57
2	69	0	2	10	18	16	8	9	12	20	68	57	64	67	68
3	70	2	0	12	17	14	7	11	14	22	70	59	66	68	70
4	79	10	12	0	5	5	6	22	26	33	82	70	78	80	82
5	87	18	17	5	0	8	10	28	31	39	87	76	83	85	87
6	84	16	14	5	8	0	8	25	28	36	74	73	80	82	84
7	77	8	7	6	10	8	0	18	22	29	78	66	73	76	77
8	78	9	11	22	28	25	18	0	3	11	59	48	55	58	59
9	81	12	14	26	31	28	22	3	0	8	56	45	52	54	56
10	89	20	22	33	39	36	29	11	8	0	48	37	44	67	68
11	2	68	70	82	87	74	78	59	56	48	0	21	9	12	13
12	3	57	59	70	76	73	66	48	45	37	2	0	7	10	11
13	11	64	66	78	83	80	73	55	52	44	9	7	0	5	6
14	13	67	68	80	85	82	76	58	54	67	12	10	5	0	2
15	15	68	70	82	87	84	77	59	56	68	13	11	6	2	0

Table 13. Matrix of distances from each dairy farm to every other dairy farm, Route C

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	0	91	132	144	144	153	148	141	135	130	127	127	124	123	124	116
2	37	0	43	55	59	76	58	33	55	55	50	56	50	49	47	42
3	80	43	0	12	16	23	16	10	12	12	24	19	18	19	18	23
4	92	55	12	0	5	18	3	13	19	24	26	28	30	30	28	34
5	96	59	16	5	0	10	8	16	23	28	32	36	36	35	34	39
6	102	76	23	18	10	0	9	15	21	26	30	33	35	34	32	37
7	95	58	16	3	8	9	0	9	16	21	24	27	26	27	26	31
8	90	33	10	13	16	15	9	0	7	12	15	18	18	18	18	23
9	92	55	12	19	23	21	16	7	0	5	8	12	11	12	11	16
10	92	55	12	24	28	26	20	12	5	0	3	6	6	72	7	12
11	88	50	24	26	32	29	24	15	8	3	0	3	3	4	6	11
12	91	56	19	28	36	33	26	18	12	6	3	0	5	6	10	14
13	87	50	18	30	36	35	26	18	11	6	3	5	0	1	4	9
14	85	49	19	30	35	34	27	18	12	7	4	6	1	0	4	8
15	82	47	18	28	34	32	26	18	11	7	6	10	4	4	0	5
16	77	42	23	34	39	37	31	23	16	12	11	14	9	8	5	0